EXPANSION IN GRAPHS AND GRAPH LIMITS

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CERTIFICATE

This is to certify that **Pokharanakar Mugdha Mahesh**, Integrated Ph.D. student in the Department of Mathematics, has completed bonafide work on the thesis entitled **'Expansion in Graphs and Graph Limits'** under my supervision and guidance.

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Pokharanakar Mugdha Mahesh

ABSTRACT

The discrete Cheeger–Buser inequality was established for graphs by Dodziuk, Alon and Milman. It states that the combinatorial expansion in graphs is equivalent to the (one-sided) spectral expansion, that is, the Cheeger constant of a graph and the gap between the two smallest eigenvalues of the Laplacian of the graph control each other. The analogous inequality, called the dual Cheeger–Buser inequality, relating the bipartiteness ratio (or the dual Cheeger constant) of a graph and the gap between 2 and the largest eigenvalue of the Laplacian of the graph was obtained by Trevisan and Bauer–Jost. We present its proof, following Trevisan's ideas, in the first chapter of the thesis. In the second chapter, we discuss the higher-order Cheeger inequalities for regular graphs, based on the work of Lee, Oveis Gharan and Trevisan. These inequalities give the relation between the k-way expansion constant of a graph and the kth smallest eigenvalue of its Laplacian. The case $k = 2$ corresponds to the discrete Cheeger–Buser inequality.

The theory of graph limits was developed by Lovász and his collaborators. Recently, Khetan and Mj introduced the notion of Cheeger constant and the Laplacian for graph limits, namely graphons and graphings, and extended the discrete Cheeger–Buser inequality for graphs to these graph limits. We define the bipartiteness ratio of graphons, and establish the dual Cheeger–Buser inequality for graphons, in the last chapter. We also show that the discrete Cheeger–Buser inequality for graphs can be recovered from it, upto a multiplicative constant.

CONTENTS

1. THE DUAL CHEEGER–BUSER INEQUALITY FOR GRAPHS

A discrete version of the Cheeger–Buser inequality on Riemmanian manifolds was established for simple graphs by Dodziuk [Dod84], Alon–Milman [AM85], and Alon [Alo86]. It gives the following relation between the second smallest eigenvalue λ of the Laplacian of a (finite undirected) graph and its Cheeger constant h.

$$\frac{h^2}{2} \leq \lambda \leq \sqrt{2h}.$$

The Cheeger constant of a graph quantifies how "well-connected" the graph is. It is well-known that λ is zero if and only if the graph is not connected. The discrete Cheeger–Buser inequality can be seen as a quantitative version of this fact.

On the other hand, it is known that the largest eigenvalue of the Laplacian of a graph is 2 if and only if the graph has a bipartite connected component. Trevisan [Tre12] and Bauer–Jost [BJ13] defined the constants which measure "bipartiteness" of a graph, namely, the bipartiteness ratio $\beta(G)$ and the dual Cheeger constant $\bar{h}$ of a graph G, respectively, and they related these constants to the gap $2-\lambda^{\max}$ between 2 and the largest eigenvalue $\lambda^{\max}$ of the Laplacian of a graph G. This relation is known as the dual Cheeger–Buser inequality. We remark that Bauer and Jost used a technique developed by Desai and Rao [DR94] in their work.

Here, we present Trevisan's proof [Tre12] of the dual Cheeger–Buser inequality for weighted graphs. We have also referred to [Tre17, Chapter 6] for some details in the proof.

1.1 Preliminaries

Let V be a finite set with $|V| = n \geq 2$. Let $\ell^2(V)$ denote the inner product space of functions $f\colon V \to \mathbb{R}$ with the inner product

$$\langle f, g\rangle \coloneqq \sum_{v\in V} f(v)g(v),$$

and the norm

$$\|f\| \coloneqq \sqrt{\langle f, f\rangle},$$

for all $f, g \in \ell^2(V)$. For any subset A of V, we denote the characteristic function of A by 1_A, and if A is the singleton set $\{a\}$, then we use the notation 1_a to denote 1_A. Let $T\colon \ell^2(V) \to \ell^2(V)$ be a self-adjoint operator with $a_{uv} \coloneqq \langle T1_v, 1_u\rangle \geq 0$ and $d_v \coloneqq \langle T1_V, 1_v\rangle > 0$ for all $u, v \in V$. Further, let $D\colon \ell^2(V) \to \ell^2(V)$ denote the linear operator defined by $(Df)(v) \coloneqq d_v f(v)$, for all $f \in \ell^2(V)$, and L denote the linear operator $I - D^{-1/2}TD^{-1/2}$ on $\ell^2(V)$, where I denotes the identity operator on $\ell^2(V)$. Note that L is a self-adjoint operator, and hence, it has n real eigenvalues. We denote by $\lambda^{\max}$, the largest eigenvalue of L. Then, we have

$$\lambda^{\max} = \max_{f\in\ell^2(V)\setminus\{0\}} \frac{\langle Lf, f\rangle}{\langle f, f\rangle},$$

and thus,

$$\begin{aligned}
2 - \lambda^{\max} &= \min_{f\in\ell^2(V)\setminus\{0\}} \frac{\langle (2I - L)f, f\rangle}{\langle f, f\rangle} \\
&= \min_{f\in\ell^2(V)\setminus\{0\}} \frac{\left\langle (I + D^{-1/2}TD^{-1/2})f, f\right\rangle}{\langle f, f\rangle} \\
&= \min_{f\in\ell^2(V)\setminus\{0\}} \frac{\left\langle (I + D^{-1/2}TD^{-1/2})(D^{1/2}f), D^{1/2}f\right\rangle}{\langle D^{1/2}f, D^{1/2}f\rangle} \\
&= \min_{f\in\ell^2(V)\setminus\{0\}} \frac{\left\langle D^{1/2}(I + D^{-1/2}TD^{-1/2})D^{1/2}f, f\right\rangle}{\langle Df, f\rangle} \\
&= \min_{f\in\ell^2(V)\setminus\{0\}} \frac{\langle (D + T)f, f\rangle}{\langle Df, f\rangle}, \qquad (1.1)
\end{aligned}$$

that is,

$$\begin{aligned}
2 - \lambda^{\max} &= \min_{f \in \ell^2(V)\setminus\{0\}} \frac{\sum_{v\in V} d_v f(v)^2 + \sum_{u,v\in V} a_{uv} f(u) f(v)}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{f \in \ell^2(V)\setminus\{0\}} \frac{\sum_{u,v\in V} a_{uv} (f(u)+f(v))^2}{2\sum_{v\in V} d_v f(v)^2}. \qquad (1.2)
\end{aligned}$$

Definition 1.1 (Bipartiteness ratio)**.** Given any linear operator $T\colon \ell^2(V) \to \ell^2(V)$ as above, the *bipartiteness ratio* β_T of T is defined by

$$\beta_T = \min_{\substack{\psi\colon V \to \{-1,0,1\} \\ \psi \neq 0}} \frac{\langle (D+T)\psi, \psi\rangle}{2\langle D\psi, \psi\rangle}.$$

Note that

$$\begin{aligned}
\beta_T &= \min_{\substack{\psi\colon V \to \{-1,0,1\} \\ \psi \neq 0}} \frac{\sum_{u,v\in V} a_{uv} (\psi(v)^2 + \psi(u)\psi(v))}{2\sum_{v\in V} d_v \psi(v)^2} \qquad (1.3) \\
&= \min_{\substack{\psi\colon V \to \{-1,0,1\} \\ \psi \neq 0}} \frac{\sum_{u,v\in V} a_{uv} (\psi(u) + \psi(v))^2}{4\sum_{v\in V} d_v \psi(v)^2}. \qquad (1.4)
\end{aligned}$$

1.2 The dual Cheeger–Buser inequality for graphs

Theorem 1.2 (The dual Cheeger–Buser inequality for graphs)**.** *For every linear operator $T\colon \ell^2(V) \to \ell^2(V)$ as above, the inequality*

$$\frac{\beta_T^2}{2} \leq 2 - \lambda^{\max} \leq 2\beta_T$$

holds.

We will use the following lemma in the proof of the above theorem.

Lemma 1.3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\mu(\Omega) > 0$, and $f\colon \Omega \to \mathbb{R}$, $g\colon \Omega \to (0,\infty)$ be integrable functions. Then there exists an $x_0 \in \Omega$ such that

$$\frac{f(x_0)}{g(x_0)} \leq \frac{\int_\Omega f(x)\,\mathrm{d}\mu(x)}{\int_\Omega g(x)\,\mathrm{d}\mu(x)}.$$

Proof. Since the function g takes only positive values and $\mu(\Omega) > 0$, we have $\int_\Omega g(x)\,\mathrm{d}\mu(x) > 0$. Further, note that

$$\int_\Omega \left(f(x) - \frac{\int_\Omega f(t)\,\mathrm{d}\mu(t)}{\int_\Omega g(t)\,\mathrm{d}\mu(t)} g(x) \right) \mathrm{d}\mu(x) = 0. \tag{1.5}$$

If $f(x) - \frac{\int_\Omega f(t)\,\mathrm{d}\mu(x)}{\int_\Omega g(t)\,\mathrm{d}\mu(t)} g(x)$ is positive for all x, then its integration over Ω is positive, as $\mu(\Omega)$ is positive, which contradicts Eq. (1.5). Hence, there is an $x_0 \in \Omega$ such that

$$f(x_0) - \frac{\int_\Omega f(x)\,\mathrm{d}\mu(x)}{\int_\Omega g(x)\,\mathrm{d}\mu(x)} g(x_0) \leq 0,$$

that is

$$\frac{f(x_0)}{g(x_0)} \leq \frac{\int_\Omega f(x)\,\mathrm{d}\mu(x)}{\int_\Omega g(x)\,\mathrm{d}\mu(x)}.$$

□

Proof of Theorem 1.2. The inequality $2 - \lambda^{\max} \leq 2\beta_T$ follows immediately from Eq. (1.1) and the definition of β_T. To get the other inequality, thanks to Eq. (1.2) and Eq. (1.3), it suffices to show that given any nonzero function $f\colon V \to \mathbb{R}$, there is a nonzero function $\psi\colon V \to \{-1, 0, 1\}$ such that

$$\frac{\sum_{u,v\in V} a_{uv}(\psi(v)^2 + \psi(u)\psi(v))}{2\sum_{v\in V} d_v \psi(v)^2} \leq \left(\frac{\sum_{u,v\in V} a_{uv}(f(u) + f(v))^2}{\sum_{v\in V} d_v f(v)^2} \right)^{\frac{1}{2}}. \tag{1.6}$$

Let $f\colon V \to \mathbb{R}$ be a nonzero function, and M denote the positive number $\max_{v\in V} |f(v)|$. For every $0 < t \leq M$, define the function $\psi_t\colon V \to \{-1, 0, 1\}$ by

$$\psi_t(v) = \begin{cases} -1 & \text{if } f(v) \leq -t, \\ 0 & \text{if } -t < f(v) < t, \\ 1 & \text{if } f(v) \geq t. \end{cases}$$

For all $0 < t \leq M$, note that the function ψ_t is nonzero. We will show using

Lemma 1.3 that Eq. (1.6) is satisfied by $\psi = \psi_{t_0}$ for some $0 < t_0 \leq M$.

Observe that

$$\begin{aligned}
&\int_0^M 2t \sum_{u,v \in V} a_{uv}(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t \\
&= \sum_{u,v \in V} a_{uv} \int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t \\
&= \sum_{\substack{u,v \in V \\ f(u)f(v) \geq 0 \\ |f(u)| \leq |f(v)|}} a_{uv} \int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t \\
&\quad + \sum_{\substack{u,v \in V \\ f(u)f(v) \geq 0 \\ |f(v)| < |f(v)|}} a_{uv} \int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t \\
&\quad + \sum_{\substack{u,v \in V \\ f(u)f(v) < 0 \\ |f(u)| \leq |f(v)|}} a_{uv} \int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t. \\
&\quad + \sum_{\substack{u,v \in V \\ f(u)f(v) < 0 \\ |f(v)| < |f(u)|}} a_{uv} \int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t
\end{aligned}$$

Let u, v be arbitrary elements of V. Consider the following cases.

Case 1: $f(u)f(v) \geq 0$ and $|f(u)| \leq |f(v)|$

Suppose that $0 \leq f(u) \leq f(v)$.

- If $0 < t \leq f(u)$, then $\psi_t(u) = \psi_t(v) = 1$.
- If $f(u) < t \leq f(v)$, then $\psi_t(u) = 0$ and $\psi_t(v) = 1$.
- If $t > f(v)$, then $\psi_t(u) = \psi_t(v) = 0$.

So, we have

$$\begin{aligned}
\int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t &= \int_0^{f(u)} 4t\,\mathrm{d}t + \int_{f(u)}^{f(v)} 2t\,\mathrm{d}t \\
&= 2f(u)^2 + f(v)^2 - f(u)^2 \\
&= f(u)^2 + f(v)^2.
\end{aligned}$$

Now, suppose that $f(v) \leq f(u) \leq 0$.
- If $0 < t \leq -f(u)$, then $\psi_t(u) = \psi_t(v) = -1$.
- If $-f(u) < t \leq -f(v)$, then $\psi_t(u) = 0$ and $\psi_t(v) = -1$.
- If $t > -f(v)$, then $\psi_t(u) = \psi_t(v) = 0$.

Thus, here also, we get

$$\int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t = f(u)^2 + f(v)^2.$$

Case 2: $f(u)f(v) \geq 0$ and $|f(v)| < |f(u)|$
Using the arguments similar to those in the above case, we obtain

$$\int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t = \int_0^{|f(v)|} 4t\,\mathrm{d}t = 2f(v)^2.$$

Case 3: $f(u)f(v) < 0$ and $|f(u)| \leq |f(v)|$
Suppose that $f(u) < 0 < f(v)$.
- If $0 < t \leq -f(u)$, then $\psi_t(u) = -1$ and $\psi_t(v) = 1$.
- If $-f(u) < t \leq f(v)$, then $\psi_t(u) = 0$ and $\psi_t(v) = 1$.
- If $t > f(v)$, then $\psi_t(u) = \psi_t(v) = 0$.

So, we have

$$\int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t = \int_{-f(u)}^{f(v)} 2t\,\mathrm{d}t = f(v)^2 - f(u)^2.$$

Now, suppose that $f(v) < 0 < f(u)$.
- If $0 < t \leq f(u)$, then $\psi_t(u) = 1$ and $\psi_t(v) = -1$.
- If $f(u) < t \leq -f(v)$, then $\psi_t(u) = 0$ and $\psi_t(v) = -1$.
- If $t > -f(v)$, then $\psi_t(u) = \psi_t(v) = 0$.

Thus, here also, we get

$$\int_0^M 2t(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t = f(v)^2 - f(u)^2.$$

Case 4: $f(u)f(v) < 0$ and $|f(v)| < |f(u)|$
Using the arguments similar to those in the above case, note that $\psi_t(v)^2 + \psi_t(u)\psi_t(v) = 0$, for all $0 < t \leq M$.

Hence, we have

$$
\begin{aligned}
&\int_0^M 2t \sum_{u,v\in V} a_{uv}(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t \\
&= \sum_{\substack{u,v\in V\\ f(u)f(v)\geq 0\\ |f(u)|\leq|f(v)|}} a_{uv}(f(u)^2+f(v)^2) + \sum_{\substack{u,v\in V\\ f(u)f(v)\geq 0\\ |f(v)|<|f(u)|}} 2a_{uv}f(v)^2 \\
&\quad + \sum_{\substack{u,v\in V\\ f(u)f(v)<0\\ |f(u)|\leq|f(v)|}} a_{uv}(f(v)^2-f(u)^2) \\
&\leq \sum_{\substack{u,v\in V\\ f(u)f(v)\geq 0}} a_{uv}(f(u)^2+f(v)^2) + \sum_{\substack{u,v\in V\\ f(u)f(v)<0}} a_{uv}\left|f(u)^2-f(v)^2\right| \\
&\leq \sum_{\substack{u,v\in V\\ f(u)f(v)\geq 0}} a_{uv}(f(u)+f(v))^2 + \sum_{\substack{u,v\in V\\ f(u)f(v)<0}} a_{uv}\left|f(u)^2-f(v)^2\right| \\
&\leq \sum_{\substack{u,v\in V\\ f(u)f(v)\geq 0}} a_{uv}(f(u)+f(v))(|f(u)|+|f(v)|) \\
&\quad + \sum_{\substack{u,v\in V\\ f(u)f(v)<0}} a_{uv}\,|f(u)+f(v)|\,|f(u)-f(v)| \\
&\leq \sum_{u,v\in V} a_{uv}\,|f(u)+f(v)|\,(|f(u)|+|f(v)|) \\
&\leq \left(\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2\right)^{\frac{1}{2}} \left(\sum_{u,v\in V} a_{uv}(|f(u)|+|f(v)|)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2\right)^{\frac{1}{2}} \left(\sum_{u,v\in V} a_{uv}(2f(u)^2+2f(v)^2)\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2\right)^{\frac{1}{2}} \left(4\sum_{u,v\in V} a_{uv}f(v)^2\right)^{\frac{1}{2}} \\
&= 2\left(\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2\right)^{\frac{1}{2}} \left(\sum_{v\in V} d_v f(v)^2\right)^{\frac{1}{2}}. \qquad (1.7)
\end{aligned}
$$

Also, note that

$$\begin{aligned}\int_0^M 2t \cdot 2\sum_{v\in V} d_v\psi_t(v)^2\,\mathrm{d}t &= \sum_{v\in V} d_v \int_0^M 4t\psi_t(v)^2\,\mathrm{d}t\\ &= \sum_{v\in V} d_v \int_0^{|f(v)|} 4t\,\mathrm{d}t\\ &= 2\sum_{v\in V} d_v f(v)^2. \qquad (1.8)\end{aligned}$$

Now using the facts that d_v is positive for all $v \in V$ and the function ψ_t is nonzero for all $t \in (0, M]$, and combining (1.7) with (1.8) yield the inequality

$$\begin{aligned}&\frac{\int_0^M 2t\sum_{u,v\in V} a_{uv}(\psi_t(v)^2 + \psi_t(u)\psi_t(v))\,\mathrm{d}t}{\int_0^M 2t\cdot 2\sum_{v\in V} d_v\psi_t(v)^2\,\mathrm{d}t}\\ &\le \frac{2\left(\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2\right)^{\frac{1}{2}}\left(\sum_{v\in V} d_v f(v)^2\right)^{\frac{1}{2}}}{2\sum_{v\in V} d_v f(v)^2}\\ &= \left(\frac{\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2}{\sum_{v\in V} d_v f(v)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, there exists a real number $t_0 \in (0, M]$ such that

$$\frac{\sum_{u,v\in V} a_{uv}(\psi_{t_0}(v)^2 + \psi_{t_0}(u)\psi_{t_0}(v))}{2\sum_{v\in V} d_v\psi_{t_0}(v)^2} \le \left(\frac{\sum_{u,v\in V} a_{uv}(f(u)+f(v))^2}{\sum_{v\in V} d_v f(v)^2}\right)^{\frac{1}{2}},$$

using Lemma 1.3, as desired. $\square$

2. HIGHER-ORDER CHEEGER INEQUALITIES

We have discussed in Chapter 1 that the discrete Cheeger–Buser inequality gives a relation between the second smallest eigenvalue of the Laplacian of a graph and its Cheeger constant (also known as the edge expansion). Lee, Oveis Gharan and Trevisan [LGT14] established an analog of this for other eigenvalues of the Laplacian of a graph. As an analog of the Cheeger constant of a graph G with vertex set V, they introduced the k-way expansion constant, for every $1 \leq k \leq |V|$. They proved that for every graph G with vertex set V and every $1 \leq k \leq |V|$, the inequality

$$\frac{\lambda_k}{2} \leq \phi_G(k) \leq O(k^2)\sqrt{\lambda_k}$$

holds, where λ_k is the kth smallest eigenvalue of the Laplacian of G and $\phi_G(k)$ is its k-way expansion constant. This is a quantitative version of the fact that the graph G has at least k connected components if and only if the kth smallest eigenvalue of its Laplacian is 0.

The above bounds have been improved, and several generalizations of the above inequality have been established. Also, various notions of expansions and higher-order Cheeger inequalities for them have been studied. For instance, see [LGT14, Liu15, AL20, MMSV24].

Here we will prove a weaker bound than the above. In the following, thinking of V as the vertex set of a weighted regular graph, and T as its adjacency matrix gives the higher-order Cheeger inequalities for weighted regular graphs. The proof follows the ideas in Trevisan's proof [Tre17].

2.1 Preliminaries

Let V be a finite set with $|V| = n \geq 2$. Let $\ell^2(V)$ denote the inner product space of functions $f\colon V \to \mathbb{R}$ with the inner product

$$\langle f, g\rangle \coloneqq \sum_{v\in V} f(v)g(v),$$

and the norm

$$\|f\| \coloneqq \sqrt{\langle f, f\rangle},$$

for all $f, g \in \ell^2(V)$. Let $T\colon \ell^2(V) \to \ell^2(V)$ be a self-adjoint operator with $a_{uv} \coloneqq \langle T1_v, 1_u\rangle \geq 0$ and $\langle T1_V, 1_u\rangle = d > 0$, for all $u, v \in V$. Let $L\colon \ell^2(V) \to \ell^2(V)$ denote the positive-semidefinite operator $I - \frac{1}{d}T$, and

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

denote the eigenvalues of L. Using the Courant–Fischer min-max theorem, for any $1 \leq k \leq n$, we have

$$\lambda_k = \min_{\substack{W \leq \ell^2(V)\\ k\text{-dimensional}}} \max_{f\in W\setminus\{0\}} \frac{\langle Lf, f\rangle}{\langle f, f\rangle}. \tag{2.1}$$

The quantity $\frac{\langle Lf,f\rangle}{\langle f,f\rangle}$ is called the *Rayleigh quotient* of f with respect to L. We denote it by $R_L(f)$.

For any $f \in \ell^2(V)$, observe that

$$\begin{aligned}
\langle Lf, f\rangle = \sum_{u\in V}(Lf)(u)f(u) &= \sum_{u\in V}\left(\left(I - \frac{1}{d}T\right)f\right)(u)f(u)\\
&= \sum_{u\in V}\left[f(u)^2 - \frac{1}{d}\sum_{v\in V} a_{uv}f(v)f(u)\right]\\
&= \frac{1}{d}\sum_{u\in V}\left[df(u)^2 - \sum_{v\in V} a_{uv}f(v)f(u)\right]\\
&= \frac{1}{d}\sum_{u\in V}\left[\sum_{v\in V} a_{uv}f(u)^2 - \sum_{v\in V} a_{uv}f(v)f(u)\right]
\end{aligned}$$

$$= \frac{1}{d} \sum_{u,v \in V} (a_{uv} f(u)^2 - a_{uv} f(v) f(u)),$$

and hence, we have

$$\begin{aligned}
\langle Lf, f \rangle &= \frac{1}{d} \sum_{u,v \in V} (a_{uv} f(u)^2 - a_{uv} f(v) f(u)) \\
&= \frac{1}{2d} \sum_{u,v \in V} (a_{uv} f(u)^2 - 2a_{uv} f(v) f(u) + a_{uv} f(v)^2) \\
&= \frac{1}{2d} \sum_{u,v \in V} a_{uv} (f(u) - f(v))^2. \qquad (2.2)
\end{aligned}$$

Definition 2.1 (k-way expansion constant)**.** For every $1 \le k \le n$ and a linear operator $T\colon \ell^2(V) \to \ell^2(V)$ as above, the *k-way expansion constant* of T, denoted by $\phi_T(k)$, is defined by

$$\phi_T(k) \coloneqq \min_{\substack{\emptyset \neq S_1, \ldots, S_k \subseteq V \\ S_1, \ldots, S_k \text{ disjoint}}} \max_{1 \le i \le k} \phi(S_i),$$

where for any nonempty subset S of V, we denote by $\phi(S)$, the *edge expansion* of S, which is defined as follows.

$$\phi(S) \coloneqq \frac{\langle T 1_{V \setminus S}, 1_S \rangle}{d|S|}.$$

Note that $\phi_T(1) = 0$, and $\phi_T(2)$ is the edge Cheeger constant of T.

Theorem 2.2. *For any opertor T as above and $1 \le k \le n$, the following inequality holds.*

$$\frac{\lambda_k}{2} \le \phi_T(k) \le O(k^{3.5}) \sqrt{\lambda_k}.$$

Note that the above inequality holds trivially for $k = 1$, and the case $k = 2$ is the well-known discrete Cheeger–Buser inequality.

2.2 Proof of the easy direction

Lemma 2.3. For any $1 \le k \le n$, the inequality $\lambda_k \le 2\phi_T(k)$ holds.

Proof. Let $S_1, \ldots, S_k$ be nonempty disjoint subsets of V such that $\phi_T(k) = \max_{1 \le i \le k} \phi(S_i)$. Let $W = \mathrm{span}\{1_{S_1}, \ldots, 1_{S_k}\}$, which is a k-dimensional subspace of $\ell^2(V)$. We will show that the Rayleigh quotient of every nonzero function in W is at most $2\phi_k(T)$, so that we are done using Eq. (2.1).

For any $1 \le i \le k$, note that

$$
\begin{aligned}
R_L(1_{S_i}) = \frac{\langle L 1_{S_i}, 1_{S_i} \rangle}{\langle 1_{S_i}, 1_{S_i} \rangle} &= \frac{\left\langle \left(I - \frac{1}{d} T\right) 1_{S_i}, 1_{S_i} \right\rangle}{\langle 1_{S_i}, 1_{S_i} \rangle} \\
&= \frac{\langle (dI - T) 1_{S_i}, 1_{S_i} \rangle}{d|S_i|} \\
&= \frac{d|S_i| - \langle T 1_{S_i}, 1_{S_i} \rangle}{d|S_i|} \\
&= \frac{\langle T 1_V, 1_{S_i} \rangle - \langle T 1_{S_i}, 1_{S_i} \rangle}{d|S_i|} \\
&= \frac{\left\langle T 1_{V \setminus S_i}, 1_{S_i} \right\rangle}{d|S_i|} \\
&= \phi(S_i) \\
&\le \phi_T(k).
\end{aligned}
$$

Then, the desired inequality follows from the following lemma using the facts that for distinct i and j, if f_i and f_j lie in the span of 1_{S_i} and 1_{S_j}, respectively, then the functions f_i and f_j are disjointly supported, and that the Rayleigh quotients are invariant under scaling. □

Lemma 2.4. Let $f_1, f_2, \ldots, f_k \in \ell^2(V)$ be disjointly supported nonzero functions. Then, we have the inequality

$$
R_L\left(\sum_{i=1}^{k} f_i\right) \le 2 \max_{1 \le i \le k} R_L(f_i).
$$

Proof. From Eq. (2.2), it follows that

$$
\left\langle L\left(\sum_{i=1}^{k} f_i\right), \sum_{i=1}^{k} f_i \right\rangle = \frac{1}{2d} \sum_{u,v \in V} a_{uv} \left(\sum_{i=1}^{k} (f_i(u) - f_i(v))\right)^2 .
$$

Let $u, v \in V$ be arbitrary. Since the functions $f_1, \ldots, f_k$ are disjointly supported, each of u and v is in the support of at most one of these functions. That is, there exist indices $j_u, l_v \in \{1, \dots, k\}$ such that for any $i \neq j_u$, we have $f_i(u) = 0$ and for any $i \neq l_v$, we have $f_i(v) = 0$. Hence, we get

$$\left(\sum_{i=1}^{k}(f_i(u) - f_i(v))\right)^2 = (f_{j_u}(u) - f_{l_v}(v))^2.$$

Now, if $j_u = l_v$, then the above equation implies that

$$\left(\sum_{i=1}^{k}(f_i(u) - f_i(v))\right)^2 = \sum_{i=1}^{k}(f_i(u) - f_i(v))^2 \leq 2\sum_{i=1}^{k}(f_i(u) - f_i(v))^2,$$

and if $j_u \neq l_v$, then

$$\begin{aligned}
\left(\sum_{i=1}^{k}(f_i(u) - f_i(v))\right)^2 &\leq 2(f_{j_u}(u))^2 + 2(f_{l_v}(v))^2 \\
&= 2[(f_{j_u}(u) - 0)^2 + (0 - f_{l_v}(v))^2] \\
&= 2\sum_{i=1}^{k}(f_i(u) - f_i(v))^2.
\end{aligned}$$

Thus, once again using Eq. (2.2), we obtain

$$\begin{aligned}
\left\langle L\left(\sum_{i=1}^{k} f_i\right), \sum_{i=1}^{k} f_i\right\rangle &\leq 2\frac{1}{2d}\sum_{u,v\in V} a_{uv}\sum_{i=1}^{k}(f_i(u) - f_i(v))^2 \\
&= 2\sum_{i=1}^{k}\frac{1}{2d}\sum_{u,v\in V} a_{uv}(f_i(u) - f_i(v))^2 \\
&= 2\sum_{i=1}^{k}\langle Lf_i, f_i\rangle.
\end{aligned}$$

Note that the functions $f_1, \dots, f_k$, being disjointly supported, are mutually orthogonal. So, we have

$$\left\langle \sum_{i=1}^{k} f_i, \sum_{i=1}^{k} f_i\right\rangle = \sum_{i=1}^{k}\langle f_i, f_i\rangle,$$

implying that

$$R_L\left(\sum_{i=1}^{k} f_i\right) = \frac{\left\langle L\left(\sum_{i=1}^{k} f_i\right), \sum_{i=1}^{k} f_i\right\rangle}{\left\langle \sum_{i=1}^{k} f_i, \sum_{i=1}^{k} f_i\right\rangle} \leq 2\frac{\sum_{i=1}^{k} \langle Lf_i, f_i\rangle}{\sum_{i=1}^{k} \langle f_i, f_i\rangle} \leq 2 \max_{1\leq i\leq k} R_L(f_i),$$

where the last inequality follows from the definition of the Rayleigh quotients along with the fact that if $a_1, \ldots, a_k$ are nonnegative real numbers and $b_1, \ldots, b_k$ are positive real numbers, then

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i \frac{a_i}{b_i} \leq \max_{1\leq i\leq k} \frac{a_i}{b_i} \sum_{i=1}^{k} b_i,$$

so that

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \leq \max_{1\leq i\leq k} \frac{a_i}{b_i}.$$

□

2.3 Proof of the difficult direction

Theorem 2.5. *For any opertor T as above and $1 \leq k \leq n$, the following inequality holds.*

$$\phi_T(k) \leq O(k^{3.5})\sqrt{\lambda_k}.$$

We will break the proof of Theorem 2.5 into several lemmas. Let us start by introducing some useful notions.

By abuse of notation, we denote the Euclidean inner product and the norm induced from it on $\mathbb{R}^k$ also, by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively.

Given functions $f_1, f_2, \ldots, f_k \in \ell^2(V)$, define the function $F\colon V \to \mathbb{R}^k$ by

$$F(v) := (f_1(v), f_2(v), \ldots, f_k(v)),$$

for all $v \in V$, which induces the following pseudo-metric *dist* on V. For any

$u, v \in V$, define

$$dist(u,v) \coloneqq \begin{cases} \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\| & \text{if } F(u), F(v) \neq 0, \\ 0 & \text{if } F(u) = F(v) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is indeed a pseudo-metric on V. Given any element v of V and a subset A of V, we define

$$dist(v, A) \coloneqq \min_{u \in A} dist(v, u).$$

Also, we extend the notion of Rayleigh quotients for the functions taking values in $\mathbb{R}^k$. Given a function $\mathbf{f}\colon V \to \mathbb{R}^k$, define the *Rayleigh quotioent* $R_L(\mathbf{f})$ of $\mathbf{f}$ with respect to L by

$$R_L(\mathbf{f}) \coloneqq \frac{\sum_{u,v \in V} a_{uv} \left\| \mathbf{f}(u) - \mathbf{f}(v) \right\|^2}{2d \sum_{v \in V} \left\| \mathbf{f}(v) \right\|^2}.$$

Now if $f_1, f_2, \ldots, f_k$ are unit vectors in $\ell^2(V)$, then we get

$$\begin{aligned} R_L(F) &= \frac{\sum_{u,v \in V} a_{uv} \left\| F(u) - F(v) \right\|^2}{2d \sum_{v \in V} \left\| F(v) \right\|^2} \\ &= \frac{\sum_{u,v \in V} a_{uv} \left\| F(u) - F(v) \right\|^2}{2d \sum_{v \in V} \sum_{1 \leq i \leq k} f_i(v)^2} \\ &= \frac{\sum_{u,v \in V} a_{uv} \left\| F(u) - F(v) \right\|^2}{2d \sum_{1 \leq i \leq k} \sum_{v \in V} f_i(v)^2} \\ &= \frac{1}{k} \frac{1}{2d} \sum_{u,v \in V} a_{uv} \left\| F(u) - F(v) \right\|^2 \qquad (2.3) \\ &= \frac{1}{k} \frac{1}{2d} \sum_{u,v \in V} a_{uv} \sum_{1 \leq i \leq k} (f_i(u) - f_i(v))^2 \\ &= \frac{1}{k} \sum_{1 \leq i \leq k} \frac{1}{2d} \sum_{u,v \in V} a_{uv} (f_i(u) - f_i(v))^2. \end{aligned}$$

Using Eq. (2.2) and the fact that each f_i has norm one, it follows that

$$R_L(F) = \frac{1}{k} \sum_{1 \le i \le k} R_L(f_i). \tag{2.4}$$

Henceforth, we assume that the functions $f_1, f_2, \ldots, f_k$, using which the function F is defined, are orthonormal.

2.3.1 Preparatory lemmas

We first state the lemmas that we will use, and come to their proofs later.

Lemma 2.6. For any $u, v \in V$ with $F(u) \neq 0$ and $F(v) \neq 0$, we have

$$\|F(v)\| \, dist(u, v) \le 2 \|F(u) - F(v)\| .$$

Lemma 2.7 (F "spreads out" vertices across $\mathbb{R}^k$)**.** Given any unit vector $\mathbf{w} \in \mathbb{R}^k$, we have

$$\sum_{v \in V} \langle F(v), \mathbf{w} \rangle^2 = 1.$$

Given any subset A of V, we call the quantity $\sum_{v \in A} \|F(v)\|^2$ the *mass* of a set A. Note that the mass of the set V equals k. For a nonempty subset R of the unit sphere in $\mathbb{R}^k$, the *diameter* of R is given by $diam(R) := \sup_{\mathbf{w}, \mathbf{z} \in R} \|\mathbf{w} - \mathbf{z}\|$ and the set $V(R)$ is defined as

$$V(R) := \left\{ v \in V : F(v) \neq 0, \frac{F(v)}{\|F(v)\|} \in R \right\}.$$

Lemma 2.8 (If R has "small" diameter, then $V(R)$ has "small" mass)**.** Any nonempty subset R of the unit sphere in $\mathbb{R}^k$, with $diam(R) < \sqrt{2}$, satisfies the inequality

$$\sum_{v \in V(R)} \|F(v)\|^2 \le \left(1 - \frac{1}{2} diam(R)^2\right)^{-2}.$$

Lemma 2.9 (Well-separated sets each with "small" mass, but "large" total mass)**.** There exist disjoint subsets $T_1, \ldots, T_m$ of V satisfying the following conditions.

(a) $\sum_{i=1}^{m} \sum_{v \in T_i} \|F(v)\|^2 \geq k - \frac{1}{4}$,

(b) If $u \in T_i$ and $v \in T_j$ with $i \neq j$, then $dist(u, v) \geq \Omega(k^{-3})$,

(c) For every $1 \leq i \leq m$, we have $\sum_{v \in T_i} \|F(v)\|^2 \leq 1 + \frac{1}{4k}$.

Lemma 2.10 (Well-separated sets each with "large" mass)**.** There exist k disjoint subsets $A_1, \ldots, A_k$ of V satisfying the following conditions.

(a) For every $1 \leq i \leq k$, we have $\sum_{v \in A_i} \|F(v)\|^2 \geq \frac{1}{2}$,

(b) If $u \in A_i$ and $v \in A_j$ with $i \neq j$, then $dist(u, v) \geq \Omega(k^{-3})$.

Lemma 2.11 (Localization)**.** Let $A_1, \ldots, A_t$ be subsets of V such that for every $i \in \{1, \ldots, t\}$, we have $\sum_{v \in A_i} \|F(v)\|^2 \geq \frac{1}{2}$, and there is a real number $\delta \in (0, 1]$ such that if $u \in A_i$ and $v \in A_j$ with $i \neq j$, then $dist(u, v) \geq \delta$. Then there exist t disjointly supported nonzero functions $g_1, \ldots, g_t \in \ell^2(V)$ such that for every $i \in \{1, \ldots, t\}$, the following inequality is satisfied.

$$R_L(g_i) \leq O(k\delta^{-2}) R_L(F).$$

Lemma 2.12. Let $f_1, \ldots, f_k$ be orthonormal functions in $\ell^2(V)$. Then there exist disjointly supported nonzero functions $g_1, \ldots, g_k \in \ell^2(V)$ such that for every $i \in \{1, \ldots, k\}$, we have

$$R_L(g_i) \leq O(k^7) \max_{1 \leq j \leq k} R_L(f_j).$$

Lemma 2.13. Given any nonzero function $g \in \ell^2(V)$, there is a subset S of its support such that $\phi(S) \leq \sqrt{2R_L(g)}$.

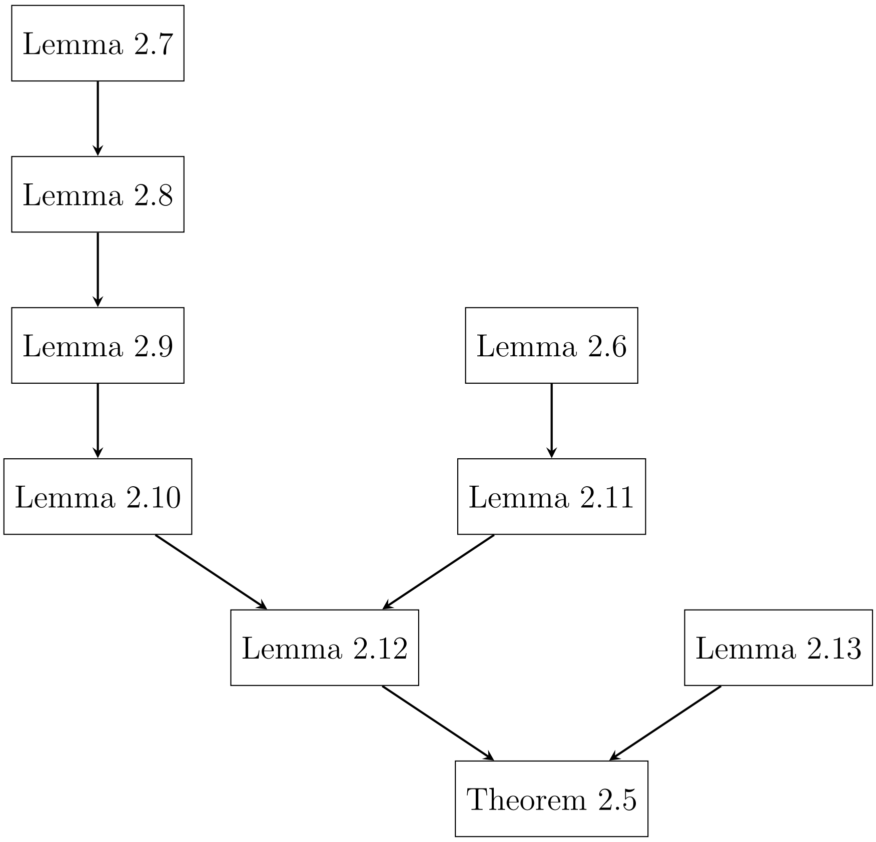

Fig. 2.1: Proof sketch of Theorem 2.5

2.3.2 Proofs of the lemmas

Proof of Lemma 2.6. Let $u, v \in V$ be arbitrary with $F(u) \neq 0$ and $F(v) \neq 0$. Then, we have

$$\begin{aligned}
\|F(v)\|\, dist(u,v) &= \|F(v)\| \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\| \\
&= \left\| F(u)\frac{\|F(v)\|}{\|F(u)\|} - F(v) \right\| \\
&= \left\| F(u)\frac{\|F(v)\|}{\|F(u)\|} - F(u) + F(u) - F(v) \right\| \\
&\leq \left\| F(u)\frac{\|F(v)\|}{\|F(u)\|} - F(u) \right\| + \|F(u) - F(v)\| \\
&= \left\| \left(\frac{\|F(v)\|}{\|F(u)\|} - 1 \right) F(u) \right\| + \|F(u) - F(v)\| \\
&= \left| \frac{\|F(v)\|}{\|F(u)\|} - 1 \right| \|F(u)\| + \|F(u) - F(v)\| \\
&= |\|F(v)\| - \|F(u)\|| + \|F(u) - F(v)\| \\
&\leq 2\, \|F(u) - F(v)\|\,,
\end{aligned}$$

where the last inequality follows from the fact that for any $\mathbf{w}, \mathbf{z} \in \mathbb{R}^k$, the inequality $|\|\mathbf{w}\| - \|\mathbf{z}\|| \leq \|\mathbf{w} - \mathbf{z}\|$ holds. □

Proof of Lemma 2.7. Define a linear map $U \colon \mathbb{R}^k \to \ell^2(V)$ by

$$(U\mathbf{w})(v) \coloneqq \langle F(v), w \rangle\,,$$

for every $\mathbf{w} \in \mathbb{R}^k$ and $v \in V$. Observe that the transpose $U^{\mathrm{t}} \colon \ell^2(V) \to \mathbb{R}^k$ of the linear map U is given by

$$U^{\mathrm{t}} f = (\langle f_1, f \rangle\,, \ldots, \langle f_k, f \rangle),$$

for every $f \in \ell^2(V)$. For $i \in \{1, \ldots, k\}$, let e_i denote the standard basis vectors in $\mathbb{R}^k$. Then, for each i, note that $U^{\mathrm{t}}Ue_i = U^{\mathrm{t}}f_i = e_i$, since the functions $f_1, \ldots, f_k$ are orthonormal. This shows that the map $U^{\mathrm{t}}U$ is the identity operator on $\mathbb{R}^k$. Hence, for any unit vector $\mathbf{w} \in \mathbb{R}^k$, we have

$$\sum_{v \in V} \langle F(v), \mathbf{w} \rangle^2 = \|U\mathbf{w}\|^2 = \left\langle U^{\mathrm{t}} U\mathbf{w}, \mathbf{w} \right\rangle = \langle \mathbf{w}, \mathbf{w} \rangle = 1.$$

□

For each $v \in V$ with $F(v) \neq 0$, define

$$\bar{F}(v) := \frac{F(v)}{\|F(v)\|}.$$

Proof of Lemma 2.8. Let R be a nonempty subset of the unit sphere in $\mathbb{R}^k$ with $diam(R) < \sqrt{2}$ and $\mathbf{w}$ be a vector in R. Then, for any $v \in V(R)$, we have $\left\|\bar{F}(v) - \mathbf{w}\right\| \leq diam(R)$. Thus, we obtain

$$\begin{aligned} diam(R)^2 \geq \left\|\bar{F}(v) - \mathbf{w}\right\|^2 &= \left\|\bar{F}(v)\right\|^2 + \|\mathbf{w}\|^2 - 2\left\langle \bar{F}(v), \mathbf{w} \right\rangle \\ &= 2 - 2\left\langle \bar{F}(v), \mathbf{w} \right\rangle, \end{aligned}$$

which implies

$$\left\langle \bar{F}(v), \mathbf{w} \right\rangle \geq 1 - \frac{1}{2} diam(R)^2,$$

and since $diam(R) < \sqrt{2}$, we conclude that

$$\frac{1}{\|F(v)\|^2} \langle F(v), \mathbf{w} \rangle^2 = \left\langle \bar{F}(v), \mathbf{w} \right\rangle^2 \geq \left(1 - \frac{1}{2} diam(R)^2\right)^2.$$

As this is true for every $v \in V(R)$, we get

$$\sum_{v \in V(R)} \langle F(v), \mathbf{w} \rangle^2 \geq \left(1 - \frac{1}{2} diam(R)^2\right)^2 \sum_{v \in V(R)} \|F(v)\|^2.$$

We arrive at the desired result using the inequality $\sum_{v \in V(R)} \langle F(v), \mathbf{w} \rangle^2 \leq 1$, which is obtained as a consequence of Lemma 2.7. □

Proof of Lemma 2.9. Set $L = \frac{1}{\sqrt{5k}}$. We tile $\mathbb{R}^k$ with cubes of the form $\prod_{i=1}^k [n_i L, n_i L + L)$, where n_i is an integer for every i. Each of these cubes has diameter equal to $\frac{1}{\sqrt{5}k}$. For every cube $C = \prod_{i=1}^{k}[n_iL, n_iL + L)$, define its *core* to be the cube

$$\tilde{C} = \prod_{i=1}^{k}\left[n_iL + \frac{L}{8k^2}, n_iL + L - \frac{L}{8k^2}\right).$$

Note that the distance between any two points in the cores of two different cubes is at least $\frac{L}{4k^2}$, which equals $\frac{1}{4\sqrt{5}k^3}$. Let $\tilde{C}_w$ denote the cube when the core $\tilde{C}$ of a cube C (as above), is shifted by $w \in \mathbb{R}^k$, that is, if $w = (w_1, \ldots, w_k)$, then we have

$$\tilde{C}_w = \prod_{i=1}^{k}\left[n_iL + \frac{L}{8k^2} + w_i, n_iL + L - \frac{L}{8k^2} + w_i\right).$$

Let $\Omega = [0, L)^k$ be a probability space equipped with the Borel σ-algebra and the probability measure $\mathbb{P}$ defined as follows. For Borel-measurable subsets $I_1, \ldots, I_k$ of $[0, L)$, define

$$\mathbb{P}(I_1 \times \cdots \times I_k) \coloneqq \frac{m(I_1) \times \cdots \times m(I_k)}{L^k},$$

where $m(B)$ denotes the Borel measure of a subset B of $[0, L)$. Fix any $v \in V$ with $F(v) \neq 0$ and let $\bar{F}(v) = (z_1, \ldots, z_k)$. Consider the set A_v defined as follows.

$$A_v \coloneqq \{w \in \Omega : \bar{F}(v) \in \tilde{C}_w \text{ for some cube } C \text{ (of the above form)}\}.$$

Observe that

$$\begin{aligned}
A_v &= \left\{(w_1, \ldots, w_k) \in \Omega \;\middle|\; \begin{array}{l} \forall\, 1 \leq i \leq k, \exists\, n_i \in \mathbb{Z} \text{ such that} \\ z_i \in \left[n_iL + \frac{L}{8k^2} + w_i, n_iL + L - \frac{L}{8k^2} + w_i\right) \end{array} \right\} \\
&= \left\{(w_1, \ldots, w_k) \in \Omega \;\middle|\; \begin{array}{l} \forall\, 1 \leq i \leq k, \exists\, n_i \in \mathbb{Z} \text{ such that} \\ w_i \in \left(z_i - n_iL - L + \frac{L}{8k^2}, z_i - n_iL - \frac{L}{8k^2}\right] \end{array} \right\} \\
&= \prod_{i=1}^{k}\left(\left(\bigcup_{n_i \in \mathbb{Z}}\left(z_i - n_iL - L + \frac{L}{8k^2}, z_i - n_iL - \frac{L}{8k^2}\right]\right) \cap [0, L)\right).
\end{aligned}$$

For every $i \in \{1, \ldots, k\}$, note that there is a unique integer n_i such that $z_i - n_iL - L + \frac{L}{8k^2}$ lies in $[0, L)$, and there is a unique integer m_i such that $z_i - m_iL - \frac{L}{8k^2}$ belongs to the interval $[0, L)$. Also, these are the only cases when the sets $J_{r_i} := \left(z_i - r_iL - L + \frac{L}{8k^2}, z_i - r_iL - \frac{L}{8k^2}\right] \cap [0, L)$, for $r_i \in \mathbb{Z}, 1 \leq i \leq k$, are nonempty. So, we have

$$\begin{aligned}
A_v &= \prod_{i=1}^{k} \Bigg(\bigg(\bigg(z_i - n_iL - L + \frac{L}{8k^2}, z_i - n_iL - \frac{L}{8k^2} \bigg] \cap [0, L) \bigg) \\
&\qquad \bigcup \bigg(\bigg(z_i - m_iL - L + \frac{L}{8k^2}, z_i - m_iL - \frac{L}{8k^2} \bigg] \cap [0, L) \bigg) \Bigg) \\
&= \prod_{i=1}^{k} (J_{n_i} \cup J_{m_i}).
\end{aligned}$$

Let $i \in \{1, \ldots, k\}$ be arbitrary. If $n_i = m_i$, then we get $J_{n_i} \cup J_{m_i} = J_{n_i}$, and

$$m(J_{n_i} \cup J_{m_i}) = z_i - n_iL - \frac{L}{8k^2} - z_i + n_iL + L - \frac{L}{8k^2} = L - \frac{L}{4k^2}.$$

On the other hand, if $n_i \neq m_i$, then we obtain $J_{n_i} = \left(z_i - n_iL - L + \frac{L}{8k^2}, L\right)$ and $J_{m_i} = \left[0, z_i - m_iL - \frac{L}{8k^2}\right]$ with $m_i = n_i + 1$, and the intervals J_{n_i} and J_{m_i} are disjoint. Hence, it follows that

$$m(J_{n_i} \cup J_{m_i}) = z_i - n_iL - L - \frac{L}{8k^2} + L - z_i + n_iL + L - \frac{L}{8k^2} = L - \frac{L}{4k^2}.$$

Thus, in any case, using Bernoulli's inequality, we have

$$\mathbb{P}(A_v) = \frac{1}{L^k} \prod_{i=1}^{k} \left(L - \frac{L}{4k^2} \right) = \left(1 - \frac{1}{4k^2} \right)^k \geq 1 - \frac{k}{4k^2} = 1 - \frac{1}{4k}.$$

Note that this is true for every $v \in V$ with $F(v) \neq 0$.

Now for each $w \in \Omega$, let $R_w := \left(\bigcup_C \tilde{C}_w \right) \cap \mathbb{S}^{k-1}$, where $\mathbb{S}^{k-1}$ denotes the unit sphere in $\mathbb{R}^k$. For every $v \in V$, define random variables $X, X_v \colon \Omega \to \mathbb{R}$ by

$$X(w) = \sum_{v \in V(R_w)} \|F(v)\|^2,$$

and

$$X_v(w) = \begin{cases} \|F(v)\|^2 1_{A_v}(w) & \text{if } F(v) \neq 0, \\ 0 & \text{if } F(v) = 0, \end{cases}$$

for every $w \in \Omega$. Then, note that $X = \sum_{v \in V} X_v$, and thus, we have

$$\begin{aligned} \mathbb{E}[X] = \sum_{v \in V} \mathbb{E}[X_v] &= \sum_{v \in V : F(v) \neq 0} \|F(v)\|^2 \mathbb{P}(A_v) \\ &\geq \left(1 - \frac{1}{4k}\right) \sum_{v \in V} \|F(v)\|^2 \\ &= \left(1 - \frac{1}{4k}\right) \sum_{v \in V} \sum_{i=1}^{k} f_i(v)^2 \\ &= \left(1 - \frac{1}{4k}\right) k \\ &= k - \frac{1}{4}. \end{aligned}$$

Now choose a point $w \in \Omega$ such that $X(w) \geq k - \frac{1}{4}$. (The existence of such a point is guaranteed by the pigeonhole principle (see [Pre20, Proposition 1.13] for instance.)) Consider those *w-shifted cores* $\tilde{C}_w$ whose intersection with the unit sphere is nonempty, and call those intersections $R_1, \ldots, R_m$. Then, we get

$$X(w) = \sum_{i=1}^{m} \sum_{v \in V(R_i)} \|F(v)\|^2 \geq k - \frac{1}{4}.$$

Also, if $u \in V(R_i)$ and $v \in V(R_j)$ with $i \neq j$, then

$$dist(u, v) = \left\| \bar{F}(u) - \bar{F}(v) \right\|,$$

and since $\bar{F}(u) \in R_i$ and $\bar{F}(v) \in R_j$, we have $dist(u, v) \geq \frac{1}{4\sqrt{5k^3}}$.

Further, since each R_i has diameter at most $\frac{1}{\sqrt{5k}}$, using Lemma 2.8, for every $i \in \{1, \ldots, m\}$, we have

$$\sum_{v \in V(R_i)} \|F(v)\|^2 \leq \left(1 - \frac{1}{10k}\right)^{-2} \leq \frac{1}{1 - \frac{1}{5k}} = \frac{5k}{5k - 1} = 1 + \frac{1}{5k - 1} \leq 1 + \frac{1}{4k}.$$

Therefore, the subsets $T_i := V(R_i)$, for $i \in \{1, \dots, m\}$, serve our purpose. $\square$

Proof of Lemma 2.10. Consider subsets $T_1, \dots, T_m$ of V as guaranteed by Lemma 2.9, and keep on merging two subsets whenever each of those subsets has mass less than $\frac{1}{2}$. Run this procedure on *new* sets as well, till at most one set having mass less than $\frac{1}{2}$ is left. Note that we get at least one set having mass $\geq \frac{1}{2}$ at the end of this process, since the above procedure keeps the *total* mass unchanged and the total mass is at least $\frac{3}{4}$. We enumerate the subsets with mass $\geq \frac{1}{2}$, left after the procedure is terminated, by $A_1, \dots, A_t$ for some $t \geq 1$.

For every i, since the mass of the subset T_i is not more than $1 + \frac{1}{4k}$, and the mass of each of the *new* sets among $A_1, \dots, A_t$ is less than 1 (as they are formed by merging two sets each of mass less than $\frac{1}{2}$), the total mass of the sets is at most $\frac{1}{2} + t\left(1 + \frac{1}{4k}\right)$. Now if t is less than k, then we have

$$\sum_{i=1}^{m} \sum_{v \in T_i} \|F(v)\|^2 \leq \frac{1}{2} + (k-1)\left(1 + \frac{1}{4k}\right) = \frac{1}{2} + k - 1 + \frac{1}{4} - \frac{1}{4k} < k - \frac{1}{4},$$

which contradicts Lemma 2.9, and hence, we conclude that $t \geq k$. Hence, the sets $A_1, \dots, A_k$ are as required. $\square$

Proof of Lemma 2.11. Let the subsets $A_1, \dots, A_t$ of V and a real number $\delta > 0$ be as in the statement of the lemma. For every $i \in \{1, \dots, t\}$, define the *smooth indicator function* τ_i of the subset A_i at any $v \in V$ as follows.

$$\tau_i(v) = \begin{cases} 0 & \text{if } dist(v, A_i) \geq \frac{\delta}{2}, \\ 1 - \frac{2}{\delta} dist(v, A_i) & \text{otherwise.} \end{cases}$$

For every i, define a function $g_i \in \ell^2(V)$ at any $v \in V$ by

$$g_i(v) := \tau_i(v) \|F(v)\| .$$

We now show that the functions $g_1, \dots, g_t$ are disjointly supported. Suppose that there are indices i, j and an element $v \in V$ such that $g_i(v)$ and $g_j(v)$ are nonzero. Then the functions τ_i and τ_j are also nonzero at v, which implies that $dist(v, A_i) < \frac{\delta}{2}$ and $dist(v, A_j) < \frac{\delta}{2}$. Hence, there exist elements $u_i \in A_i$ and $u_j \in A_j$ such that $dist(v, u_i) < \frac{\delta}{2}$ and $dist(v, u_j) < \frac{\delta}{2}$. Now it follows from the triangle inequality for the pseudo-metric $dist$ that

$$dist(u_i, u_j) \leq dist(u_i, v) + dist(v, u_j) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and then the hypothesis forces i to be equal to j. Thus, the functions $g_1, \ldots, g_t$ are disjointly supported.

Moreover, for every $i \in \{1, \ldots, t\}$, we have

$$\langle g_i, g_i \rangle = \sum_{v \in V} \tau_i^2(v) \left\| F(v) \right\|^2 \geq \sum_{v \in A_i} \tau_i^2(v) \left\| F(v) \right\|^2 = \sum_{v \in A_i} \left\| F(v) \right\|^2 \geq \frac{1}{2}.$$

In particular, each g_i is a nonzero function and its Rayleigh quotient is well-defined.

Now fix an arbitrary element i from the set $\{1, \ldots, t\}$. For any $u, v \in V$, we will prove that $|g_i(v) - g_i(u)| \leq \|F(v) - F(u)\| \left(1 + \frac{4}{\delta}\right)$. If $F(u)$ or $F(v)$ is zero, then this inequality holds trivially using the fact that $\tau_i(w) \leq 1$ for all $w \in V$. Assume now that both $F(u)$ and $F(v)$ are nonzero. Note that

$$\begin{aligned}
&|g_i(v) - g_i(u)| \\
&= |\tau_i(v) \|F(v)\| - \tau_i(u) \|F(u)\|| \\
&= |\tau_i(v) \|F(v)\| - \tau_i(v) \|F(u)\| + \tau_i(v) \|F(u)\| - \tau_i(u) \|F(u)\|| \\
&\leq |\tau_i(v) \|F(v)\| - \tau_i(v) \|F(u)\|| + |\tau_i(v) \|F(u)\| - \tau_i(u) \|F(u)\|| \\
&= \tau_i(v) |\|F(v)\| - \|F(u)\|| + \|F(u)\| |\tau_i(v) - \tau_i(u)| \\
&\leq \|F(v) - F(u)\| + \|F(u)\| |\tau_i(v) - \tau_i(u)| .
\end{aligned}$$

We claim that $|\tau_i(v) - \tau_i(u)| \leq \frac{2}{\delta} |dist(v, A_i) - dist(u, A_i)|$. If we have the inequalities $dist(v, A_i) \geq \frac{\delta}{2}$ and $dist(u, A_i) \geq \frac{\delta}{2}$, then clearly the claim holds. On the other hand, if $dist(v, A_i) < \frac{\delta}{2}$ and $dist(u, A_i) < \frac{\delta}{2}$, then observe that

$$|\tau_i(v) - \tau_i(u)| = \left| \left(1 - \frac{2}{\delta} dist(v, A_i)\right) - \left(1 - \frac{2}{\delta} dist(u, A_i)\right) \right|$$

$$= \frac{2}{\delta} \left| dist(v, A_i) - dist(u, A_i) \right| .$$

Otherwise, if $dist(u, A_i) < \frac{\delta}{2}$ and $dist(v, A_i) \geq \frac{\delta}{2}$, then we have

$$\begin{aligned}
|\tau_i(v) - \tau_i(u)| &= 1 - \frac{2}{\delta} dist(u, A_i) \\
&\leq \frac{2}{\delta} dist(v, A_i) - \frac{2}{\delta} dist(u, A_i) \\
&= \frac{2}{\delta} \left| dist(v, A_i) - dist(u, A_i) \right| ,
\end{aligned}$$

and similarly, the claim holds if $dist(v, A_i) < \frac{\delta}{2}$ and $dist(u, A_i) \geq \frac{\delta}{2}$. Then using the triangle inequality for $dist$, it follows that $|\tau_i(v) - \tau_i(u)| \leq \frac{2}{\delta} dist(v, u)$.

Hence, for any $u, v \in V$, using Lemma 2.6, we obtain

$$\begin{aligned}
|g_i(v) - g_i(u)| &\leq \|F(v) - F(u)\| + \frac{2}{\delta} \|F(u)\| \, dist(v, u) \\
&\leq \|F(v) - F(u)\| \left(1 + \frac{4}{\delta}\right) ,
\end{aligned}$$

and thus, the numerator of the Rayleigh quotient of g_i is

$$\begin{aligned}
\langle Lg_i, g_i \rangle &= \frac{1}{2d} \sum_{u,v \in V} a_{uv} (g_i(u) - g_i(v))^2 && \text{(using Eq. (2.2))} \\
&\leq \left(1 + \frac{4}{\delta}\right)^2 \frac{1}{2d} \sum_{u,v \in V} a_{uv} \|F(v) - F(u)\|^2 \\
&= \left(1 + \frac{4}{\delta}\right)^2 k R_L(F). && \text{(using Eq. (2.3))}
\end{aligned}$$

This proves that the Rayleigh quotient of g_i is

$$\begin{aligned}
R_L(g_i) = \frac{\langle Lg_i, g_i \rangle}{\langle g_i, g_i \rangle} &\leq 2 \left(1 + \frac{4}{\delta}\right)^2 k R_L(F) \\
&= 2 \left(1 + \frac{8}{\delta} + \frac{16}{\delta^2}\right) k R_L(F) \\
&\leq 50 \frac{k}{\delta^2} R_L(F),
\end{aligned}$$

where we have used the fact that δ lies in the interval $(0,1]$ in the last inequality. $\square$

Proof of Lemma 2.12. From the proof of Lemma 2.9 and Lemma 2.10, note that $\delta = \frac{1}{4\sqrt{5}k^3}$ works in the statement of Lemma 2.11. Combining this fact with Lemma 2.10 and Lemma 2.11, we obtain disjointly supported nonzero functions $g_1, \ldots, g_k \in \ell^2(V)$ such that the inequality $R_L(g_i) \leq O(k^7) R_L(F)$ holds for every $i \in \{1, \ldots, k\}$. The desired inequality follows from Eq. (2.4) and the fact that the average of k real numbers is at most as large as their maximum. $\square$

Proof of Lemma 2.13. Let $g \colon V \to \mathbb{R}$ be a nonzero function. We will prove that there is a real number $t_0 \in [0, \max_{u \in V} g(u)^2)$ such that the inequality

$$\phi(\{v \in V \mid g(v)^2 > t_0\}) \leq \sqrt{2R_L(g)}$$

holds. It is enough to prove this, as the set $\{v \in V \mid g(v)^2 > t\}$ is a nonempty subset of the support of g for every $t \in [0, \max_{u \in V} g(u)^2)$.

Let us denote the number $\max_{u \in V} g(u)^2$ by M, and for any $t \in [0, M)$, denote by S_t the set $\{v \in V \mid g(v)^2 > t\}$. Note that

$$\int_0^M d\,|S_t| \,\mathrm{d}t = d \sum_{v \in V} \int_0^M 1_{S_t}(v) \,\mathrm{d}t = d \sum_{v \in V} \int_0^{g(v)^2} 1 \,\mathrm{d}t = d \sum_{v \in V} g(v)^2,$$

and that

$$\begin{aligned}
&\int_0^M \left\langle T 1_{V \setminus S_t}, 1_{S_t} \right\rangle \,\mathrm{d}t \\
&= \sum_{u \in V} \sum_{v \in V} \int_0^M a_{uv} 1_{S_t}(u) 1_{V \setminus S_t}(v) \,\mathrm{d}t \\
&= \sum_{\substack{u,v \in V \\ g(v)^2 < g(u)^2}} \int_{g(v)^2}^{g(u)^2} a_{uv} \,\mathrm{d}t \\
&= \sum_{\substack{u,v \in V \\ g(v)^2 < g(u)^2}} a_{uv} (g(u)^2 - g(v)^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\sum_{\substack{u,v\in V\\ g(v)^2<g(u)^2}} a_{uv}(g(u)^2 - g(v)^2) + \sum_{\substack{u,v\in V\\ g(u)^2<g(v)^2}} a_{vu}(g(v)^2 - g(u)^2)\right)\\
&= \frac{1}{2}\sum_{u,v\in V} a_{uv}\left|g(u)^2 - g(v)^2\right| \qquad \text{(since } a_{uv} = a_{vu} \text{ for all } u,v\in V\text{)}\\
&= \frac{1}{2}\sum_{u,v\in V} a_{uv}\left|g(u) - g(v)\right|\left|g(u) + g(v)\right|\\
&\le \frac{1}{2}\left(\sum_{u,v\in V} a_{uv}(g(u) - g(v))^2\right)^{\frac{1}{2}}\left(\sum_{u,v\in V} a_{uv}(g(u) + g(v))^2\right)^{\frac{1}{2}}\\
&\qquad\qquad\text{(using the Cauchy–Schwarz inequality)}\\
&\le \frac{1}{2}\left(\sum_{u,v\in V} a_{uv}(g(u) - g(v))^2\right)^{\frac{1}{2}}\left(\sum_{u,v\in V} a_{uv}(2g(u)^2 + 2g(v)^2)\right)^{\frac{1}{2}}\\
&= \frac{\sqrt{2}}{2}\left(\sum_{u,v\in V} a_{uv}(g(u) - g(v))^2\right)^{\frac{1}{2}}\left(2d\sum_{v\in V} g(v)^2\right)^{\frac{1}{2}},
\end{aligned}
$$

and thus, we have

$$
\frac{\int_0^M \left\langle T1_{V\setminus S_t}, 1_{S_t}\right\rangle \,\mathrm{d}t}{\int_0^M d\left|S_t\right| \,\mathrm{d}t} \le \frac{\left(\sum_{u,v\in V} a_{uv}(g(u) - g(v))^2\right)^{\frac{1}{2}}}{\left(d\sum_{v\in V} g(v)^2\right)^{\frac{1}{2}}} = \sqrt{2R_L(g)}.
$$

Now using Lemma 1.3, we conclude that there exists a real number $t_0 \in [0, M)$ such that the following inequality holds.

$$
\frac{\left\langle T1_{V\setminus S_{t_0}}, 1_{S_{t_0}}\right\rangle}{d\left|S_{t_0}\right|} \le \frac{\int_0^M \left\langle T1_{V\setminus S_t}, 1_{S_t}\right\rangle \,\mathrm{d}t}{\int_0^M d\left|S_t\right| \,\mathrm{d}t} \le \sqrt{2R_L(g)}.
$$

Therefore, the subset S_{t_0} of the support of the function g satisfies the inequality $\phi(S_{t_0}) \le \sqrt{2R_L(g)}$. $\square$

2.3.3 Proof of Theorem 2.5

Proof of Theorem 2.5. Let $f_1, \ldots, f_k$ be orthonormal eigenfunctions corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$, respectively, of the operator L. Then Lemma 2.12 guarantees the existence of disjointly supported nonzero functions $g_1, \ldots, g_k \in \ell^2(V)$ such that for every $i \in \{1, \ldots, k\}$, we have

$$R_L(g_i) \leq O(k^7) \max_{1 \leq j \leq k} R_L(f_j).$$

For every $j \in \{1, \ldots, k\}$, note that $R_L(f_j) = \lambda_j$, and therefore, the Rayleigh quotient of each g_i is at most $O(k^7)\lambda_k$. Now applying Lemma 2.13 to every nonzero function g_i, we get a nonempty subset S_i of its support such that $\phi(S_i) \leq \sqrt{2R_L(g_i)} \leq O(k^{3.5})\sqrt{\lambda_k}$. Hence, we have nonempty disjoint subsets $S_1, \ldots, S_k$ of V such that

$$\max_{1 \leq i \leq k} \phi(S_i) \leq O(k^{3.5})\sqrt{\lambda_k}.$$

Now the desired inequality follows from the above inequality and the very definition of the k-way expansion constant of T. $\square$

3. THE DUAL CHEEGER–BUSER INEQUALITY FOR GRAPHONS

Lovász and his collaborators [LS06, BCL$^+$06, BCL$^+$08] developed the theory of graph limits, through both algebraic and analytic perspectives. They studied graphons and graphings, which arise as limits of convergent sequences of graphs and bounded degree graphs, respectively. Several results regarding graphons and graphings are discussed quite extensively in the book by Lovász [Lov12]. The theory of graph limits has found lots of connections with many other branches of mathematics, including extremal graph theory, probability theory, higher-order Fourier analysis, ergodic theory, number theory, group theory, representation theory, category theory, the limit theory of metric spaces, and numerous applications in other subjects like computer science, network theory and statistical physics.

Various notions in the context of graphs have been extended to graph limits, for instance, homomorphism densities [LS06], Szemerédi's regularity lemma [LS07], independent sets, cliques, and colorings [HR20], and tilings [HHP21]. Khetan and Mj [KM24] established the analogs of the discrete Cheeger–Buser inequality for graphs in the case of graphons and graphings. Given a connected graphon W, having Cheeger constant h_W and the bottom of the spectrum of its Laplacian λ_W, they proved that

$$\frac{h_W^2}{8} \leq \lambda_W \leq 2h_W.$$

They also showed that the Cheeger–Buser inequality for regular graphs can be recovered from this inequality for graphons.

In this chapter, we introduce the notion of bipartiteness ratio in the context of graphons. Also, we establish the dual Cheeger–Buser inequality for graphons, which relates the gap between 2 and the *top of the spectrum* of the Laplacian of a graphon with its bipartiteness ratio. We have discussed the dual Cheeger–Buser inequality for graphs in Chapter 1. Our result is its analog for graphons. We prove the following result obtained in the preprint [Pok25].

Theorem 3.1 (The dual Cheeger–Buser inequality for graphons)**.** *Let W be a connected graphon, β_W denote its bipartiteness ratio and $\lambda_W^{\max}$ denote the top of the spectrum of its Laplacian. Then the following inequality holds.*

$$\frac{\beta_W^2}{2} \leq 2 - \lambda_W^{\max} \leq 2\beta_W.$$

3.1 Preliminaries

In the following, by a measurable subset, we mean a Lebesgue measurable subset, and we denote the Lebesgue measure on $I = [0,1]$ by μ_L.

A funcion $W\colon I^2 \to I$ is called a *graphon* if W is a Lebesgue measurable function which is symmetric, that is, $W(x,y) = W(y,x)$ for all $(x,y) \in I^2$. We say that a graphon W is *connected* if $\int_{A\times A^c} W > 0$ for every measurable subset A of I with $0 < \mu_L(A) < 1$.

Let W be a connected graphon. For every measurable subset A of I and S of I^2, define

$$\nu(A) \coloneqq \int_{A\times I} W(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \text{and} \qquad \eta(S) \coloneqq \int_S W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then ν and η are measures on I and I^2, respectively. Note that the $\mathbb{R}$-vector space $L^2(I,\nu)$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_v$, given by

$$\langle f,g\rangle_v = \int_{I^2} f(x)g(x)W(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

for all $f,g \in L^2(I,\nu)$. We denote the restriction of the measure η to the measurable subsets of the set $E = \{(x,y) \in I^2 : y > x\}$ also by η, and denote the inner product on the $\mathbb{R}$-Hilbert space $L^2(E,\eta)$ by $\langle\cdot,\cdot\rangle_e$, which is given by

$$\langle f,g\rangle_e = \int_0^1\int_x^1 f(x,y)g(x,y)W(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

for all $f,g\in L^2(E,\eta)$. The norms induced by the inner products $\langle\cdot,\cdot\rangle_v$ and $\langle\cdot,\cdot\rangle_e$ are denoted by $\|\cdot\|_v$ and $\|\cdot\|_e$, respectively.

Given any function $f\in L^2(I,\nu)$, define $df(x,y) := f(y)-f(x)$, for all $(x,y)\in I^2$. Then it is proved in [KM24, Lemma 3.3] that the map $d\colon L^2(I,\nu)\to L^2(E,\eta)$ which maps $f\in L^2(I,\nu)$ to the function $df|_E\in L^2(E,\eta)$ is a bounded linear operator. Let $d^*\colon L^2(E,\eta)\to L^2(I,\nu)$ denote the adjoint of the operator d, and define the *Laplacian* Δ_W of W by $\Delta_W = d^*d$, which is a bounded linear operator on the space $L^2(I,\nu)$.

Given a graphon W, for all $x\in I$, the *degree* of x is defined by

$$d_W(x) := \int_I W(x,y)\,\mathrm{d}y.$$

If W is a connected graphon, then $\eta(I^2)=\int_I d_W(x)\,\mathrm{d}x$ is positive, and thus, d_W is positive μ_L-a.e. In that case, for every $f\in L^2(I,\nu)$ and $x\in I$ with $d_W(x)\neq 0$, it is shown in [KM24, Section 3.2] that

$$(\Delta_W f)(x) = f(x) - \frac{1}{d_W(x)}(T_W f)(x),$$

where the linear operator $T_W\colon L^2(I,\nu)\to L^2(I,\nu)$ is defined by

$$(T_W f)(x) = \int_I W(x,y)f(y)\,\mathrm{d}y.$$

For the sake of brevity, we will write $\Delta_W = I - \frac{1}{d_W}T_W$, where I denotes the identity operator on $L^2(I,\nu)$, by abuse of notation.

Definition 3.2 (Top of the spectrum)**.** Given a conncted graphon W, the *top of the spectrum* of its Laplacian Δ_W, denoted by $\lambda_W^{\max}$, is defined by

$$\lambda_W^{\max} := \sup_{f\in L^2(I,\nu)\setminus\{0\}} \frac{\langle\Delta_W f,f\rangle_v}{\langle f,f\rangle_v}.$$

Note that

$$\lambda_W^{\max} = \sup_{f \in L^2(I,\nu)\setminus\{0\}} \frac{\|df\|_e^2}{\|f\|_v^2},$$

and it follows from the proof of [KM24, Lemma 3.3] that $\lambda_W^{\max} \leq 4$. In fact, this bound improves to 2, similar to that in the case of graphs, as shown in the following lemma.

Lemma 3.3. For any $f \in L^2(I,\nu)$, the inequality $\|df\|_e \leq \sqrt{2}\,\|f\|_v$ holds, and consequently, we have $\lambda_W^{\max} \leq 2$.

Proof. Let $f \in L^2(I,\nu)$ be arbitrary. Then we have

$$\|df\|_e^2 = \int_E (df)^2 W = \int_0^1 \int_x^1 (f(y) - f(x))^2 W(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Using the fact that the graphon W is symmetric, it follows that

$$\begin{aligned}\int_0^1 \int_x^1 (f(y) - f(x))^2 W(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_y^1 (f(x) - f(y))^2 W(y,x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_y^1 (f(x) - f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{E'} (df)^2 W,\end{aligned}$$

where $E' \coloneqq \{(x,y) \in I^2 : y < x\}$. Now, as the function df is identically zero on the diagonal of I^2, observe that

$$\int_{I^2} (df)^2 W = \int_E (df)^2 W + \int_{E'} (df)^2 W = 2\,\|df\|_e^2.$$

Hence, we get

$$\begin{aligned}\|df\|_e^2 &= \frac{1}{2} \int_0^1 \int_0^1 (f(x) - f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2} \int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2} \int_0^1 \int_0^1 f(y)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_0^1 \int_0^1 |f(x)|\,|f(y)|\,W(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

that is,

$$\|df\|_e^2 = \|f\|_v^2 + \int_0^1 \int_0^1 |f(x)|\,|f(y)|\,W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now since the function f lies in $L^2(I,\nu)$, the functions

$$(x,y) \mapsto |f(x)|\sqrt{W(x,y)} \quad \text{and} \quad (x,y) \mapsto |f(y)|\sqrt{W(x,y)},$$

defined on I^2, are in $L^2(I^2)$. Then the Cauchy–Schwarz inequality implies that

$$\begin{aligned}
&\int_0^1 \int_0^1 |f(x)|\,|f(y)|\,W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \left(\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_0^1 \int_0^1 f(y)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&= \|f\|_v^2,
\end{aligned}$$

from which we conclude that $\|df\|_e^2 \leq 2\,\|f\|_v^2$. □

3.2 Bipartiteness ratio of graphons

Given any graph with vertex set V, a nonempty subset S of V and a bipartition $\{L, R\}$ of S, Trevisan considered the ratio of the number of edges incident on S which "fail to be cut" by the partition $\{L, R\}$ to the total number of edges incident on S, and defined the bipartiteness ratio of the graph to be the minimum of such ratios over all nonempty subsets and their partitions. We refer to Section 3.5 for the precise definition. We extend this definition to graphons.

Definition 3.4 (Bipartiteness ratio)**.** The *bipartiteness ratio* of a connected graphon W, denoted by β_W, is defined by

$$\beta_W \coloneqq \inf_{\substack{L,R\subseteq I \\ \text{measurable} \\ \mu_L(L\cup R)>0 \\ L\cap R=\emptyset}} \beta_W(L,R),$$

where for every measurable disjoint subsets L and R of I with $\mu_L(L \cup R) > 0$, we have

$$\beta_W(L, R) = \frac{2\eta(L \times L) + 2\eta(R \times R) + \eta((L \cup R) \times (L \cup R)^c)}{2\eta((L \cup R) \times I)}.$$

Since the graphon W is connected, the above quantity is well-defined.

We will write $\lambda_W^{\max}$, β_W and $\beta_W(L, R)$ as $\lambda^{\max}$, $\beta(L, R)$ and β, respectively, when there is no room for confusion.

Khetan and Mj [KM24, Lemma 3.2] proved that the Cheeger constant of any connected graphon is bounded above by $\frac{1}{2}$, using the strong mixing property of the doubling map. We follow the same arguments to prove that the bipartiteness ratio of connected graphons is also bounded above by $\frac{1}{2}$.

For the sake of completeness we define the notion of strong mixing and state its characterization that we will use here.

Definition 3.5 (Strong mixing)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A measurable function $T \colon \Omega \to \Omega$ is called *strong mixing* if it is a measure preserving transformation, that is, $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$, and for all measurable subsets A, $B \in \mathcal{A}$, the function T satisfies

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

It is well known that the doubling map $S \colon I \to I$, defined by

$$S(x) := \begin{cases} 2x & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 2x - 1 & \text{if } \dfrac{1}{2} < x \le 1, \end{cases}$$

is strong mixing. Then it follows that the function $T \colon I^2 \to I^2$, defined by $T = S \times S$, is also strong mixing. Here I and I^2 are endowed with the Lebesgue measure.

Lemma 3.6 (Strong mixing property)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then a measure preserving transformation $T \colon \Omega \to \Omega$ is strong mixing if and only if for all f, $g \in L^2(\mu)$, we have

$$\lim_{n\to\infty} \int_\Omega (f \circ T^n) g \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu \int_\Omega g \,\mathrm{d}\mu.$$

We denote the characteristic function of a set A by 1_A.

Lemma 3.7. For every connected graphon W, the inequality $\beta_W \leq \frac{1}{2}$ holds.

Proof. Let S denote the doubling map on I, as defined above, T denote the map $S \times S$, and η_L denote the Lebesgue measure on I^2. Set $L = \left[0, \frac{1}{2}\right]$ and $R = \left(\frac{1}{2}, 1\right]$. Using the strong mixing property for T, we get

$$\lim_{n\to\infty} \int_{I^2} (1_{L\times L} \circ T^n) W \,\mathrm{d}\eta_L = \left(\int_{I^2} 1_{L\times L} \,\mathrm{d}\eta_L\right)\left(\int_{I^2} W \,\mathrm{d}\eta_L\right) = \frac{\eta(I^2)}{4},$$

$$\lim_{n\to\infty} \int_{I^2} (1_{R\times R} \circ T^n) W \,\mathrm{d}\eta_L = \left(\int_{I^2} 1_{R\times R} \,\mathrm{d}\eta_L\right)\left(\int_{I^2} W \,\mathrm{d}\eta_L\right) = \frac{\eta(I^2)}{4},$$

$$\lim_{n\to\infty} \int_{I^2} (1_{(L\cup R)\times(L\cup R)^c} \circ T^n) W \,\mathrm{d}\eta_L = 0, \quad \text{(since the set } (L\cup R)^c \text{ is empty)}$$

and

$$\lim_{n\to\infty} \int_{I^2} (1_{(L\cup R)\times I} \circ T^n) W \,\mathrm{d}\eta_L = \left(\int_{I^2} 1_{(L\cup R)\times I} \,\mathrm{d}\eta_L\right)\left(\int_{I^2} W \,\mathrm{d}\eta_L\right) = \eta(I^2).$$

For every $n \geq 1$, let L_n and R_n denote the measurable subsets $S^{-n}(L)$ and $S^{-n}(R)$ of I, respectively. Since L and R are disjoint, so are the sets L_n and R_n for all n. Also, the fact that S is measure preserving ensures that $\mu_L(L_n \cup R_n) > 0$. Observe that

$$\begin{aligned}
&\beta_W(L_n, R_n) \\
&= \frac{2\eta(L_n \times L_n) + 2\eta(R_n \times R_n) + \eta((L_n \cup R_n) \times (L_n \cup R_n)^c)}{2\eta((L_n \cup R_n) \times I)} \\
&= \frac{2\int_{I^2} 1_{L_n\times L_n} W \,\mathrm{d}\eta_L + 2\int_{I^2} 1_{R_n\times R_n} W \,\mathrm{d}\eta_L + \int_{I^2} 1_{(L_n\cup R_n)\times(L_n\cup R_n)^c} W \,\mathrm{d}\eta_L}{2\int_{I^2} 1_{(L_n\cup R_n)\times I} W \,\mathrm{d}\eta_L}.
\end{aligned}$$

Taking limit as n tends to ∞ in the above, and using the fact that

$$1_{A\times B} \circ T^n = 1_{T^{-n}(A\times B)} = 1_{S^{-n}(A)\times S^{-n}(B)},$$

for all $n \geq 1$ and subsets A, B of I, it follows that

$$\lim_{n\to\infty} \beta_W(L_n, R_n) = \frac{\frac{\eta(I^2)}{2} + \frac{\eta(I^2)}{2}}{2\eta(I^2)} = \frac{1}{2}.$$

Now since for all n, we have $\beta_W \leq \beta_W(L_n, R_n)$, it follows that $\beta_W \leq \frac{1}{2}$. $\square$

Remark 3.8. In fact, the bound in the above lemma is sharp as can be seen from the following example. If W is a nonzero constant graphon, then for any disjoint measurable subsets L and R of I with $\mu_L(L \cup R) > 0$, we have

$$\begin{aligned}
\beta_W(L,R) &= \frac{2\eta(L\times L) + 2\eta(R\times R) + \eta((L\cup R)\times(L\cup R)^c)}{2\eta((L\cup R)\times I)} \\
&= \frac{2\mu_L(L)^2 + 2\mu_L(R)^2 + (\mu_L(L)+\mu_L(R))(1-(\mu_L(L)+\mu_L(R)))}{2(\mu_L(L)+\mu_L(R))} \\
&= \frac{1}{2} + \frac{2\mu_L(L)^2 + 2\mu_L(R)^2 - (\mu_L(L)+\mu_L(R))^2}{2(\mu_L(L)+\mu_L(R))} \\
&= \frac{1}{2} + \frac{(\mu_L(L)-\mu_L(R))^2}{2(\mu_L(L)+\mu_L(R))} \\
&\geq \frac{1}{2},
\end{aligned}$$

and hence, using Lemma 3.7, we conclude that the bipartiteness ratio of W is equal to $\frac{1}{2}$.

3.3 The dual Cheeger–Buser inequality for graphons

3.3.1 The dual Buser inequality

Here we establish an upper bound for $2-\lambda^{\max}$, by obtaining a characterization of β in terms of functions taking values in $\{-1, 0, 1\}$, extending Trevisan's idea to graphons.

Lemma 3.9. For every connected graphon W, the inequality $2 - \lambda^{\max} \leq 2\beta$ holds.

Proof. Note that

$$\begin{aligned}
&2 - \lambda^{\max} \\
&= 2 - \sup_{f \in L^2(I,\nu)\setminus\{0\}} \frac{\|df\|_e^2}{\|f\|_v^2} \\
&= 2 - \sup_{f \in L^2(I,\nu)\setminus\{0\}} \frac{\int_0^1\int_0^1 (f(x)-f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y} \\
&= \inf_{f \in L^2(I,\nu)\setminus\{0\}} \left(2 - \frac{\int_0^1\int_0^1 (f(x)-f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}\right) \\
&= \inf_{f \in L^2(I,\nu)\setminus\{0\}} \frac{4\int_{I^2} f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{I^2}(f(x)-f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_{I^2} f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}.
\end{aligned}$$

The numerator in the above expression is

$$\begin{aligned}
&4\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^1\int_0^1 (f(x)-f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 3\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^1\int_0^1 f(y)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_0^1\int_0^1 f(x)f(y) W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\int_0^1\int_0^1 f(x)f(y) W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^1\int_0^1 f(y)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_0^1\int_0^1 f(x)f(y) W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Therefore, it follows that

$$2 - \lambda^{\max} = \inf_{f \in L^2(I,\nu)\setminus\{0\}} \frac{\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{3.1}$$

In order to show that $2-\lambda^{\max} \leq 2\beta$, we now proceed to obtain an expression for β in terms of functions defined on I.

Given any disjoint measurable subsets L and R of I with $\mu_L(L \cup R) > 0$, define a function $f\colon I \to \{-1,0,1\}$ for every $x \in I$ as follows.

$$f(x) = \begin{cases} -1 & \text{if } x \in L, \\ 1 & \text{if } x \in R, \\ 0 & \text{if } x \notin L \cup R. \end{cases}$$

Then f is a nonzero function in $L^2(I,\nu)$, and we have

$$\beta_W(L,R) = \frac{\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{4\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{3.2}$$

To see this, using the definition of the function f, observe that

$$\begin{aligned}
&\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{L\cup R} \int_{L\cup R} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{L\cup R} \int_{(L\cup R)^c} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{(L\cup R)^c} \int_{L\cup R} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{(L\cup R)^c} \int_{(L\cup R)^c} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{L\cup R} \int_{L\cup R} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\eta((L\cup R)\times(L\cup R)^c) \\
&= \int_L \int_L (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_L \int_R (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+ \int_R \int_L (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_R \int_R (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ 2\eta((L\cup R)\times(L\cup R)^c) \\
= {}& 4\eta(L\times L) + 4\eta(R\times R) + 2\eta((L\cup R)\times(L\cup R)^c),
\end{aligned}
$$

and we also have

$$
\begin{aligned}
&\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{L\cup R} \int_I f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{(L\cup R)^c} \int_I f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{L\cup R} \int_I W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \eta((L\cup R)\times I).
\end{aligned}
$$

On the other hand, given a nonzero function $f\colon I\to\{-1,0,1\}$ in $L^2(I,\nu)$, the sets $L=f^{-1}(-1)$ and $R=f^{-1}(1)$ are disjoint measurable subsets of I with $\mu_L(L\cup R)>0$ such that Eq. (3.2) holds. Hence, we conclude that

$$
\beta_W = \inf_{\substack{f\colon I\to\{-1,0,1\}\\ f\in L^2(I,\nu)\setminus\{0\}}} \frac{\int_0^1\int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{4\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{3.3}
$$

Now combine Eq. (3.1) and Eq. (3.3) to get the inequality $2-\lambda^{\max}\le 2\beta$. $\square$

3.3.2 The dual Cheeger inequality

We obtain a lower bound on $2-\lambda^{\max}$ with the help of some lemmas. The following lemma allows us to work with just essentially bounded functions instead of all L^2 functions while dealing with $\lambda^{\max}$. It is inspired from the analogous lemma in the work of Khetan and Mj [KM24, Lemma 5.4].

Lemma 3.10. Given a connected graphon W, we have

$$
\lambda^{\max} = \sup_{f\in L^\infty(I,\nu)\setminus\{0\}} \frac{\|df\|_e^2}{\|f\|_v^2},
$$

and consequently, the equality

$$2 - \lambda^{\max} = \inf_{f \in L^\infty(I,\nu)\setminus\{0\}} \frac{\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}$$

holds.

Proof. Since the measure space I is ν-finite, it is clear from the inclusion $L^\infty(I,\nu) \subseteq L^2(I,\nu)$ that $\lambda^{\max}$ is an upper bound of the set

$$\left\{ \frac{\|df\|_e^2}{\|f\|_v^2} : f \in L^\infty(I,\nu) \setminus \{0\} \right\}.$$

Let $\varepsilon > 0$ be arbitrary. It is enough to show, for the first part, that there is a function $g \in L^\infty(I,\nu) \setminus \{0\}$ such that the inequality

$$\lambda^{\max} - \varepsilon < \frac{\|dg\|_e^2}{\|g\|_v^2}$$

holds. The definition of $\lambda^{\max}$ guarantees the existence of a function $f \in L^2(I,\nu) \setminus \{0\}$ with

$$\lambda^{\max} - \frac{\varepsilon}{2} < \frac{\|df\|_e^2}{\|f\|_v^2}.$$

So, we are done once we find $g \in L^\infty(I,\nu) \setminus \{0\}$ satisfying

$$\frac{\|df\|_e^2}{\|f\|_v^2} - \frac{\|dg\|_e^2}{\|g\|_v^2} \leq \frac{\varepsilon}{2}. \tag{3.4}$$

If the function df is zero, then (3.4) holds by taking g to be the constant function 1. Suppose df is nonzero, and define $M = \min\{\|f\|_v, \|df\|_e\}$, which is a positive real number. As the space $L^\infty(I,\nu)$ is dense in $L^2(I,\nu)$, there exists a function $g \in L^\infty(I,\nu)$ such that $\|g-f\|_v < \varepsilon' M$, where $\varepsilon' = \min\left\{\frac{1}{\sqrt{2}}, \frac{(\sqrt{2}-1)^2\varepsilon}{16(\sqrt{2}+1)}\right\}$. Then, we get the inequality

$$(1-\varepsilon')\|f\|_v \leq \|f\|_v - \varepsilon' M < \|g\|_v < \|f\|_v + \varepsilon' M \leq (1+\varepsilon')\|f\|_v. \tag{3.5}$$

This ensures that g is a nonzero function. Using Lemma 3.3, since we have

$$\|d(g-f)\|_e < \sqrt{2}\varepsilon' M \leq \sqrt{2}\varepsilon' \|df\|_e ,$$

it follows that

$$(1-\sqrt{2}\varepsilon') \|df\|_e < \|dg\|_e < (1+\sqrt{2}\varepsilon') \|df\|_e . \tag{3.6}$$

Using (3.5) and the fact that $\varepsilon' \leq 1$, we get

$$\|f\|_v^2 \|g\|_v^2 \geq (1-\varepsilon')^2 \|f\|_v^4 , \tag{3.7}$$

and combining (3.5), (3.6) and Lemma 3.3 gives us that

$$\|df\|_e^2 \|g\|_v^2 - \|dg\|_e^2 \|f\|_v^2 \leq ((1+\varepsilon')^2 - (1-\sqrt{2}\varepsilon')^2) \|f\|_v^2 \|df\|_e^2 . \tag{3.8}$$

Now using (3.7), (3.8) and the fact that $\|df\|_e^2 \leq 2\|f\|_v^2$ along with the definition of ε', we arrive at the desired inequality (3.4).

In order to prove the second part, repeat the arguments used to obtain Eq. (3.1) by replacing $L^2(I,\nu)$ with $L^\infty(I,\nu)$. □

In the following lemma, we estimate the integrals of certain "suitable" functions so that those estimates combined with Lemma 1.3 give an upper bound for β in terms of $\lambda^{\max}$. This follows ideas in Trevisan's proof (see [Tre12, Section 3.2] and [Tre17, Chapter 6]) of the dual Cheeger inequality for graphs.

Lemma 3.11. Let f be an arbitrary element of $L^2(I,\nu)$. For every $t > 0$, let L_t and R_t denote the sets $f^{-1}((-\infty,-t])$ and $f^{-1}([t,\infty))$, respectively. Then the following inequalities hold.

$$\begin{aligned}&\int_0^\infty 2t\,[2\eta(L_t\times L_t) + 2\eta(R_t\times R_t) + \eta((L_t\cup R_t)\times(L_t\cup R_t)^c)]\,\mathrm{d}t \\ &\leq 2\left(\int_{I^2} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{I^2} f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} ,\end{aligned} \tag{3.9}$$

and

$$\int_0^\infty 2t\,[2\eta((L_t \cup R_t) \times I)]\,\mathrm{d}t = 2\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.10}$$

Proof. Using the Fubini–Tonelli theorem, note that

$$\begin{aligned}
&\int_0^\infty 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t \\
&= \int_0^\infty 2t \left[\int_{I^2} \left(2 \cdot 1_{L_t \times L_t} + 2 \cdot 1_{R_t \times R_t} + 1_{(L_t \cup R_t) \times (L_t \cup R_t)^c}\right) W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right] \mathrm{d}t \\
&= \int_{I^2} \left(\int_0^\infty 2t(2 \cdot 1_{L_t \times L_t}(x,y) + 2 \cdot 1_{R_t \times R_t}(x,y) 1_{(L_t \cup R_t) \times (L_t \cup R_t)^c}(x,y))\,\mathrm{d}t\right) \\
&\qquad W(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Define the sets

$$\begin{aligned}
A_1 &= \{(x,y) \in I^2 : 0 \le f(x) \le f(y)\}, \\
A_2 &= \{(x,y) \in I^2 : 0 \le f(y) < f(x)\}, \\
A_3 &= \{(x,y) \in I^2 : f(x) < f(y) \le 0\}, \\
A_4 &= \{(x,y) \in I^2 : f(y) \le f(x) \le 0\}, \\
A_5 &= \{(x,y) \in I^2 : f(x)f(y) < 0, |f(y)| < |f(x)|\}.
\end{aligned}$$

Now given any $(x,y) \in I^2$, observe that

$$1_{L_t \times L_t}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A_3 \text{ and } t \in (0, -f(y)], \\ 1 & \text{if } (x,y) \in A_4 \text{ and } t \in (0, -f(x)], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$1_{R_t \times R_t}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A_1 \text{ and } t \in (0, f(x)], \\ 1 & \text{if } (x,y) \in A_2 \text{ and } t \in (0, f(y)], \\ 0 & \text{otherwise,} \end{cases}$$

and that

$$1_{(L_t\cup R_t)\times(L_t\cup R_t)^c}(x,y)=\begin{cases}1 & \text{if } (x,y)\in A_2 \text{ and } t\in(f(y),f(x)],\\ 1 & \text{if } (x,y)\in A_3 \text{ and } t\in(-f(y),-f(x)],\\ 1 & \text{if } (x,y)\in A_5 \text{ and } t\in(|f(y)|,|f(x)|],\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we get

$$\begin{aligned}&\int_0^1\int_0^1\left[\int_0^\infty 2t\,(2\cdot 1_{L_t\times L_t}(x,y))\,\mathrm{d}t\right]W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_3}\left(\int_0^{-f(y)}4t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_4}\left(\int_0^{-f(x)}4t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_3}2f(y)^2W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_4}2f(x)^2W(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\int_0^1\left[\int_0^\infty 2t\,(2\cdot 1_{R_t\times R_t}(x,y))\,\mathrm{d}t\right]W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_1}\left(\int_0^{f(x)}4t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_2}\left(\int_0^{f(y)}4t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_1}2f(x)^2W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_2}2f(y)^2W(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\int_0^1\left[\int_0^\infty 2t\left(1_{(L_t\cup R_t)\times(L_t\cup R_t)^c}(x,y)\right)\mathrm{d}t\right]W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_2}\left(\int_{f(y)}^{f(x)}2t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_3}\left(\int_{-f(y)}^{-f(x)}2t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{A_5}\left(\int_{|f(y)|}^{|f(x)|}2t\,\mathrm{d}t\right)W(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{A_2}(f(x)^2-f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_{A_3}(f(x)^2-f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$+\int_{A_5} (f(x)^2 - f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, we finally have

$$\begin{aligned}
&\int_0^\infty 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t \\
=& \int_{A_1} 2f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_2} (f(x)^2 + f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{A_3} (f(x)^2 + f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_4} 2f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{A_5} (f(x)^2 - f(y)^2)W(x,y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

which is finite, as the function f lies in $L^2(I,\nu)$. Now since each of the above integrands is less than or equal to $|f(x)+f(y)|\,(|f(x)|+|f(y)|)W(x,y)$, it follows that

$$\begin{aligned}
&\int_0^\infty 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t \\
\le& \int_0^1 \int_0^1 |f(x)+f(y)|\,(|f(x)|+|f(y)|)W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
\le& \left(\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\left(\int_0^1 \int_0^1 (|f(x)|+|f(y)|)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad \text{(using the Cauchy–Schwarz inequality in } L^2(I^2)\text{)} \\
\le& \left(\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\left(\int_0^1 \int_0^1 (2f(x)^2+2f(y)^2) W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad \text{(for real numbers } a,b,\ (|a|+|b|)^2 \le 2a^2+2b^2\text{)} \\
=& 2\left(\int_{I^2} (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{I^2} f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}$$

Similar calculations give us

$$
\begin{aligned}
&\int_0^\infty 2t\,(2\eta((L_t \cup R_t) \times I))\,\mathrm{d}t \\
&= \int_0^1 \int_0^1 \left(\int_0^\infty 4t \cdot 1_{(L_t \cup R_t) \times I}(x,y)\,\mathrm{d}t \right) W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_0^1 \left(\int_0^{|f(x)|} 4t\,\mathrm{d}t \right) W(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2 \int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

as desired. □

Note that the measures μ_L and ν on I are absolutely continuous with respect to each other, and hence we have $L^\infty(I,\mu_L) = L^\infty(I,\nu)$. Henceforth, we will denote these spaces by $L^\infty(I)$. Also, observe that if f lies in $L^\infty(I)$, then its essential suprema with respect to both the measures are the same. We denote them by $\|f\|_\infty$.

Proof of Theorem 3.1. We have already proved one of the inequalities in Lemma 3.9. For the other inequality, thanks to Lemma 3.10, it suffices to show that for every nonzero function $f \in L^\infty(I)$, there exist disjoint measurable subsets L and R of I with $\mu_L(L \cup R) > 0$ such that the following inequality holds.

$$
\beta(L,R) \le \left(\frac{\int_0^1 \int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1 \int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y} \right)^{\frac{1}{2}}.
$$

Let f be any nonzero function in $L^\infty(I)$. For every $t \in (0, \|f\|_\infty)$, the sets L_t and R_t, as defined in Lemma 3.11, are disjoint measurable subsets of I with $\mu_L(L_t \cup R_t) > 0$. Also, for $t > \|f\|_\infty$, the sets L_t and R_t have measure zero. This implies that

$$
\int_0^\infty 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t
$$

$$= \int_0^{\|f\|_\infty} 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t$$

and that

$$\int_0^{\infty} 2t\,[2\eta((L_t \cup R_t) \times I)]\,\mathrm{d}t = \int_0^{\|f\|_\infty} 2t\,[2\eta((L_t \cup R_t) \times I)]\,\mathrm{d}t,$$

where the integrand $4t\eta((L_t \cup R_t) \times I)$ is positive for every $t \in (0, \|f\|_\infty)$. Hence, using Lemma 3.11, we arrive at the inequality

$$\begin{aligned}
&\frac{\int_0^{\|f\|_\infty} 2t\,[2\eta(L_t \times L_t) + 2\eta(R_t \times R_t) + \eta((L_t \cup R_t) \times (L_t \cup R_t)^c)]\,\mathrm{d}t}{\int_0^{\|f\|_\infty} 2t\,[2\eta((L_t \cup R_t) \times I)]\,\mathrm{d}t} \\
&\leq \frac{2\left(\int_0^1\int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}}{2\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y} \\
&= \left(\frac{\int_0^1\int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now Lemma 1.3 guarantees that there is a $t_0 \in (0, \|f\|_\infty)$ such that

$$\beta(L_{t_0}, R_{t_0}) \leq \left(\frac{\int_0^1\int_0^1 (f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\int_0^1 f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}\right)^{\frac{1}{2}},$$

which completes the proof. □

3.4 Bipartite graphons

Khetan and Mj [KM24, Section 7.3] gave necessary and sufficient conditions for a graphon to be conncted, under some suitable hypothesis. In this section, we characterize bipartite graphons in terms of the top of the spectrum of their Laplacians and their bipartiteness ratios, under the same hypothesis. We start by recalling the definition of bipartite graphons.

Definition 3.12 (Bipartite graphon)**.** A graphon W is said to be *bipartite* if there exist disjoint measurable subsets L and R of I such that $L \cup R = I$

and W is zero almost everywhere on $L \times L$ and $R \times R$ (with respect to the Lebesgue measure on I^2).

We will use [KM24, Lemma 7.11] which states that if W is a connected graphon and the function d_W is bounded below by a positive real number, then the operator $\frac{1}{d_W}T_W \colon L^2(I,\nu) \to L^2(I,\nu)$ is compact.

Lemma 3.13. Let W be a connected graphon such that d_W is bounded below by a positive real number. Then $\lambda_W^{\max}$ is an eigenvalue of the Laplacian Δ_W of W.

Proof. Since the Laplacian of W is a self-adjoint bounded linear operator on the Hilbert space $L^2(I,\nu)$, its top of the spectrum $\lambda_W^{\max}$ is its approximate eigenvalue, using [Lim96, Theorem 27.5(a)]. Then $1-\lambda_W^{\max}$ is an approximate eigenvalue of the operator $I - \Delta_W = I - \left(I - \frac{1}{d_W}T_W\right) = \frac{1}{d_W}T_W$. Note that $\frac{1}{d_W}T_W$ is a compact operator by [KM24, Lemma 7.11]. We know that every nonzero approximate eigenvalue of a compact operator on a Hilbert space is its eigenvalue (see [Lim96, Lemma 28.4(a)] for instance). Hence, in our case, $1-\lambda_W^{\max}$ is an eigenvalue of $\frac{1}{d_W}T_W$. Then it follows that $\lambda_W^{\max}$ is an eigenvalue of Δ_W. $\square$

The following lemma characterizes bipartite graphons.

Lemma 3.14. Let W be a connected graphon such that d_W is bounded below by a positive real number. Then the following statements are equivalent.

1. $\beta_W = 0$.
2. $\lambda_W^{\max} = 2$.
3. The graphon W is bipartite.

Proof. The implication (1) $\implies$ (2) follows from the dual Buser inequality (Lemma 3.9), and the implication (3) $\implies$ (1) is a direct consequence of the definitions of β_W and bipartite graphons. Now we prove that (2) $\implies$ (3).

Suppose that $\lambda_W^{\max} = 2$. Then Lemma 3.13 ensures that 2 is an eigenvalue of Δ_W. Let $f \in L^2(I,\nu)$ be its corresponding eigenfunction. Then the arguments similar to those used to prove Eq. (3.1) yield the equation

$$\frac{\int_{I^2}(f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}{2\int_{I^2} f(x)^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y}=0,$$

and hence,

$$\int_{I^2}(f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y=0. \tag{3.11}$$

Denote the sets $f^{-1}(-\infty,0)$ and $f^{-1}(0,\infty)$ by L and R, respectively. Then Eq. (3.11) gives that

$$\int_{(L\cup R)\times(L\cup R)^c}(f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y=0.$$

For any $(x,y)\in(L\cup R)\times(L\cup R)^c$, since we have $(f(x)+f(y))^2=f(x)^2>0$, it follows that W is zero almost everywhere on $(L\cup R)\times(L\cup R)^c$. Now since W is connected, this implies that the Lebesgue measure of either $L\cup R$ or its complement is zero. But the fact that the function f is nonzero forces $(L\cup R)^c$ to have measure zero. Let L' denote the set $L\cup(L\cup R)^c$. Note that L' and R are disjoint measurable subsets of I, and their union is I. We are done once we show that W is zero almost everywhere on $L'\times L'$ and $R\times R$. It follows from Eq. (3.11) that

$$\int_{L\times L}(f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y=0,$$

and

$$\int_{R\times R}(f(x)+f(y))^2 W(x,y)\,\mathrm{d}x\,\mathrm{d}y=0.$$

For all $(x,y)\in L\times L$, the quantity $(f(x)+f(y))^2$ is positive, and therefore, W is zero almost everywhere on $L\times L$, and hence also on $L'\times L'$, as $(L\cup R)^c$ has measure zero. Similarly, it follows that W is zero almost everywhere on $R\times R$. $\square$

3.5 Graphs and the associated graphons

Let V denote the set $\{1, \dots, n\}$ with $n \geq 2$, and $w \colon V \times V \to I$ be a symmetric function, that is, $w(i,j) = w(j,i)$ for all $i, j \in V$. The pair $G = (V, w)$ is called a *weighted graph*. We will denote $w(i,j)$ by w_{ij}, for all $i, j \in V$. The weighted graph G is said to be *loopless* if $w_{ii} = 0$ for all $i \in V$. For any subsets A, B of V, define

$$e_G(A, B) = \sum_{i \in A, j \in B} w_{ij}.$$

In the following, we always assume that G is *connected*, which means that for any nonempty proper subset A of V, $e_G(A, A^c)$ is positive. For any $i \in V$, the *volume* of i is defined by $\mathrm{vol}(i) = \sum_{j \in V} w_{ij}$. Note that $\mathrm{vol}(i)$ is positive for all i, since the graph G is connected.

Let $\ell^2(V)$ denote the Hilbert space of all functions from V to $\mathbb{R}$, equipped with the inner product

$$\langle f_1, f_2 \rangle \coloneqq \sum_{i \in V} f_1(i) f_2(i),$$

for every $f_1, f_2 \in \ell^2(V)$. The *Laplacian* $\Delta_G \colon \ell^2(V) \to \ell^2(V)$ of the weighted graph G is a linear operator defined by

$$(\Delta_G g)(i) \coloneqq g(i) - \frac{1}{\sqrt{\mathrm{vol}(i)}} \sum_{j \in V} \frac{g(j) w_{ij}}{\sqrt{\mathrm{vol}(j)}},$$

for every $g \in \ell^2(V)$ and $i \in V$. It is a self-adjoint operator, since the function w is symmetric. Then the largest eigenvalue $\lambda_G^{\max}$ of the Laplacian Δ_G is given by

$$\lambda_G^{\max} = \sup_{g \in \ell^2(V) \setminus \{0\}} \frac{\langle \Delta_G g, g \rangle}{\langle g, g \rangle} = \sup_{g \in \ell^2(V) \setminus \{0\}} \frac{\left\langle \Delta_G(\sqrt{D} g), \sqrt{D} g \right\rangle}{\left\langle \sqrt{D} g, \sqrt{D} g \right\rangle},$$

where $\sqrt{D}$ is an invertible operator on $\ell^2(V)$ defined by

$$(\sqrt{D} h)(i) = \sqrt{\mathrm{vol}(i)}\, h(i),$$

for all $h \in \ell^2(V)$ and $i \in V$. It is easy to check that

$$\lambda_G^{\max} = \sup_{g \in \ell^2(V) \setminus \{0\}} \frac{\sum_{i,j \in V} (g(i) - g(j))^2 w_{ij}}{2 \sum_{i,j \in V} g(i)^2 w_{ij}}.$$

The *bipartiteness ratio* β_G of the weighted graph G is defined as follows.

$$\beta_G = \min_{\substack{A,B \subseteq V \\ A \cup B \neq \emptyset \\ A \cap B = \emptyset}} \frac{2e_G(A,A) + 2e_G(B,B) + e_G(A \cup B, (A \cup B)^c)}{2e_G(A \cup B, V)}.$$

Now given a weighted graph $G = (V, w)$, it can be viewed as the graphon, called the *associated graphon* W_G of G, defined as below. For each $1 \leq i < n$, denote the interval $[\frac{i-1}{n}, \frac{i}{n})$ by P_i, and $[\frac{n-1}{n}, 1]$ by P_n. Note that $\{P_i \times P_j : 1 \leq i, j \leq n\}$ forms a partition of I^2. For any $1 \leq i, j \leq n$ and $(x, y) \in P_i \times P_j$, define $W_G(x, y) \coloneqq w_{ij}$.

We will show that the connectedness of G implies the connectedness of W_G, so that we can talk about the Laplacian and the bipartiteness ratio of W_G.

Lemma 3.15. If $G = (V, w)$ is a connected weighted graph, then the associated graphon W_G of G is also connected.

Proof. Let A be a measurable subset of I with $0 < \mu_L(A) < 1$. Then the sets

$$S_1 = \{i \in V : \mu_L(A \cap P_i) > 0\} \quad \text{and} \quad S_2 = \{j \in V : \mu_L(A^c \cap P_j) > 0\}$$

are nonempty, and the inclusions $S_1^c \subseteq S_2$ and $S_2^c \subseteq S_1$ hold. Further, observe that

$$\begin{aligned}\int_{A \times A^c} W = \sum_{i,j \in V} \int_{(A \cap P_i) \times (A^c \cap P_j)} W &= \sum_{i \in S_1, j \in S_2} \int_{(A \cap P_i) \times (A^c \cap P_j)} w_{ij} \\ &= \sum_{i \in S_1, j \in S_2} \mu_L(A \cap P_i) \mu_L(A^c \cap P_j) w_{ij} \\ &\geq m_A e_G(S_1, S_2),\end{aligned}$$

where $m_A = \min\{\mu_L(A \cap P_i)\mu_L(A^c \cap P_j) : i \in S_1, j \in S_2\} > 0$. Now if both S_1 and S_2 are equal to V, then we have $e_G(S_1, S_2) \geq e_G(\{1\}, \{1\}^c) > 0$ as the graph G is connected. Otherwise, if S_1 or S_2 is not V, then we get $e_G(S_1, S_2) \geq e_G(S_1, S_1^c) > 0$ or $e_G(S_1, S_2) \geq e_G(S_2^c, S_2) > 0$, respectively. Thus, in any case, $m_A e_G(S_1, S_2)$ and hence $\int_{A \times A^c} W$ is positive, showing that the associated graphon W_G is connected. □

3.5.1 Top of the spectrum of graphs and the associated graphons

The arguments in the following lemma are similar to that in [KM24, Section 4.2].

Lemma 3.16. Given any loopless, connected weighted graph $G = (V, w)$, we have $\lambda_{W_G}^{\max} = \lambda_G^{\max}$.

Proof. Let $g \colon V \to \mathbb{R}$ be any nonzero function. It gives rise to a nonzero function $g' \in L^\infty(I)$, defined for any $x \in P_i$ with $1 \leq i \leq n$, by $g'(x) = g(i)$, that satisfies

$$\begin{aligned}
\lambda_{W_G}^{\max} &\geq \frac{\int_0^1 \int_0^1 (g'(x) - g'(y))^2 W_G(x, y) \,\mathrm{d}x \,\mathrm{d}y}{2 \int_0^1 \int_0^1 g'(x)^2 W_G(x, y) \,\mathrm{d}x \,\mathrm{d}y} \\
&= \frac{\sum_{i,j \in V} \int_{P_i \times P_j} (g'(x) - g'(y))^2 W_G(x, y) \,\mathrm{d}x \,\mathrm{d}y}{2 \sum_{i,j \in V} \int_{P_i \times P_j} g'(x)^2 W_G(x, y) \,\mathrm{d}x \,\mathrm{d}y} \\
&= \frac{\sum_{i,j \in V} (g(i) - g(j))^2 w_{ij}}{2 \sum_{i,j \in V} g(i)^2 w_{ij}}.
\end{aligned}$$

Hence, we get the inequality $\lambda_{W_G}^{\max} \geq \lambda_G^{\max}$.

On the other hand, given a nonzero function $f \in L^\infty(I)$, define the function $F \colon V \to \mathbb{R}$ by $F(i) = \int_{P_i} f(x) \,\mathrm{d}x$, for every $i \in V$. Then the definition of $\lambda_G^{\max}$ gives the inequality

$$\frac{1}{2} \sum_{i,j \in V} (F(i) - F(j))^2 w_{ij} \leq \lambda_G^{\max} \sum_{i,j \in V} F(i)^2 w_{ij},$$

that is,

$$\sum_{i,j\in V} F(i)^2 w_{ij} - \sum_{i,j\in V} F(i)F(j)w_{ij} \leq \lambda_G^{\max} \sum_{i,j\in V} F(i)^2 w_{ij},$$

and hence, we have

$$-\sum_{i,j\in V} F(i)F(j)w_{ij} \leq (\lambda_G^{\max} - 1) \sum_{i,j\in V} F(i)^2 w_{ij}. \tag{3.12}$$

Now note that

$$\begin{aligned}
\sum_{i,j\in V} F(i)F(j)w_{ij} &= \sum_{i,j\in V} \left(\int_{P_i} f(x)\,\mathrm{d}x\right)\left(\int_{P_j} f(y)\,\mathrm{d}y\right) w_{ij} \\
&= \sum_{i,j\in V} \int_{P_i\times P_j} f(x)f(y)W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_0^1 f(x)f(y)W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

and that

$$\begin{aligned}
\sum_{i,j\in V} F(i)^2 w_{ij} &= \sum_{i,j\in V} \left(\int_{P_i} f(x)\,\mathrm{d}x\right)^2 w_{ij} \\
&\leq \frac{1}{n} \sum_{i,j\in V} \left(\int_{P_i} f(x)^2\,\mathrm{d}x\right) w_{ij} \\
&\qquad\qquad \text{(using the Cauchy–Schwarz inequality)} \\
&= \sum_{i,j\in V} \left(\int_{P_i} f(x)^2\,\mathrm{d}x\right)\left(\int_{P_j} 1\,\mathrm{d}y\right) w_{ij} \\
&= \sum_{i,j\in V} \int_{P_i\times P_j} f(x)^2 W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_0^1 f(x)^2 W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus, using the fact that the largest eigenvalue of the Laplacian of a loopless graph is ≥ 1 [Chu97, Lemma 1.7(ii)], (3.12) becomes

$$-\int_0^1\int_0^1 f(x)f(y)W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq (\lambda_G^{\max}-1)\int_0^1\int_0^1 f(x)^2 W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which implies

$$\frac{\|df\|_e^2}{\|f\|_v^2} = 1 - \frac{\int_0^1\int_0^1 f(x)f(y)W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\int_0^1 f(x)^2 W_G(x,y)\,\mathrm{d}x\,\mathrm{d}y} \leq 1 + (\lambda_G^{\max}-1) = \lambda_G^{\max}.$$

This proves that $\lambda_{W_G}^{\max} \leq \lambda_G^{\max}$, as desired. □

Remark 3.17. Combining Lemma 3.14 and Lemma 3.16 with the fact that a connected graph is bipartite if and only if the largest eigenvalue of its Laplacian is 2, we conclude that a connected graph is bipartite if and only if its associated graphon is bipartite.

3.5.2 Bipartiteness ratio of graphs and the associated graphons

Let $G = (V, w)$ be a connected weighted graph. Recall that $V = \{1, \dots, n\}$ with $n \geq 2$. We now obtain a characterization for the bipartite ratio β_{W_G} of the associated graphon W_G of G in terms of certain elements of I^n, analogous to the notion of the fractional Cheeger constant introduced by Khetan and Mj [KM24].

For every $\alpha = (\alpha_1, \dots, \alpha_n), \gamma = (\gamma_1, \dots, \gamma_n) \in I^n$ with $0 < \alpha + \gamma \leq 1$, where $0 = (0, \dots, 0), 1 = (1, \dots, 1) \in I^n$, define

$$\tilde{\beta}_G(\alpha,\gamma) := \frac{\sum_{i,j\in V}[2\alpha_i\alpha_j + 2\gamma_i\gamma_j + (\alpha_i+\gamma_i)(1-(\alpha_j+\gamma_j))]w_{ij}}{2\sum_{i,j\in V}(\alpha_i+\gamma_i)w_{ij}}.$$

Note that $\sum_{i,j\in V}(\alpha_i+\gamma_i)w_{ij} = \sum_{i,j\in V}(\alpha_i+\gamma_i)\mathrm{vol}(i)$, which is positive, since $\mathrm{vol}(i) > 0$ for all i.

Lemma 3.18. Given a connected weighted graph $G = (V, w)$, we have

$$\beta_{W_G} = \inf_{\substack{\alpha,\gamma\in I^n \\ 0<\alpha+\gamma\leq 1}} \tilde{\beta}_G(\alpha,\gamma). \tag{3.13}$$

Proof. It suffices to prove that the sets

$$A = \{\tilde{\beta}_G(\alpha,\gamma) : \alpha,\gamma \in I^n, 0 < \alpha+\gamma \leq 1\},$$

and

$$B = \{\beta_{W_G}(L,R) : L, R \text{ are disjoint measurable subsets of } I, \mu_L(L\cup R) > 0\}$$

are equal. Let $\alpha = (\alpha_1, \dots, \alpha_n), \gamma = (\gamma_1, \dots, \gamma_n)$ be elements of I^n with $0 < \alpha + \gamma \leq 1$. Then, observe that the sets

$$L = \bigcup_{i\in V} \left(\frac{i-1}{n}, \frac{i-1+\alpha_i}{n}\right) \quad \text{and} \quad R = \bigcup_{j\in V} \left(\frac{j-\gamma_j}{n}, \frac{j}{n}\right)$$

are disjoint measurable subsets of I and $\mu_L(L\cup R) > 0$, and thus, $\beta_{W_G}(L,R)$ lies in the set B. We will show that $\beta_{W_G}(L,R) = \tilde{\beta}_G(\alpha,\gamma)$, so that we can conclude that $\tilde{\beta}_G(\alpha,\gamma)$ also belongs to the set B. For that, note that

$$\begin{aligned}
&2\eta(L\times L) + 2\eta(R\times R) + \eta((L\cup R)\times(L\cup R)^c) \\
&= \sum_{i,j\in V} [2\mu_L(L\cap P_i)\mu_L(L\cap P_j) + 2\mu_L(R\cap P_i)\mu_L(R\cap P_j) \\
&\quad + \mu_L((L\cup R)\cap P_i)\mu_L((L\cup R)^c\cap P_j)]w_{ij} \\
&= \sum_{i,j\in V} \left[2\frac{\alpha_i}{n}\frac{\alpha_j}{n} + 2\frac{\gamma_i}{n}\frac{\gamma_j}{n} + \left(\frac{\alpha_i+\gamma_i}{n}\right)\left(\frac{1-(\alpha_j+\gamma_j)}{n}\right)\right] w_{ij} \\
&= \frac{1}{n^2}\sum_{i,j\in V} [2\alpha_i\alpha_j + 2\gamma_i\gamma_j + (\alpha_i+\gamma_i)(1-(\alpha_j+\gamma_j))]w_{ij},
\end{aligned}$$

and that

$$2\eta((L\cup R)\times I) = 2\sum_{i,j\in V} \mu_L((L\cup R)\cap P_i)\mu_L(P_j)w_{ij}$$

$$
\begin{aligned}
&= \frac{2}{n} \sum_{i,j\in V} \left(\frac{\alpha_i + \gamma_i}{n}\right) w_{ij} \\
&= \frac{2}{n^2} \sum_{i,j\in V} (\alpha_i + \gamma_i) w_{ij}.
\end{aligned}
$$

Combining the above two equations gives us that $\beta_{W_G}(L, R) = \tilde{\beta}_G(\alpha, \gamma)$, and this proves that A is a subset of B. To obtain the other inclusion, start with disjoint measurable subsets L and R of I with $\mu_L(L \cup R) > 0$, and set $\alpha_i = n\mu_L(L \cap P_i)$ and $\beta_j = n\mu_L(R \cap P_j)$, for every $i, j \in V$. Then the above calculations show that the elements $\alpha = (\alpha_1, \ldots, \alpha_n), \gamma = (\gamma_1, \ldots, \gamma_n)$ of I^n are such that $0 < \alpha + \gamma \leq 1$ and $\beta_{W_G}(L, R) = \tilde{\beta}_G(\alpha, \gamma)$, implying that B is a subset of A. □

From the arguments similar to that in the proof of Lemma 3.18, and the fact that $e_G(A, B) = n^2 \eta(\bigcup_{i\in A, j\in B} P_i \times P_j)$, for all subsets A, B of V, it follows that

$$
\beta_G = \min_{\substack{\alpha,\gamma\in\{0,1\}^n \\ 0<\alpha+\gamma\leq 1}} \tilde{\beta}_G(\alpha, \gamma). \tag{3.14}
$$

The next lemma, which is similar to [KM24, Lemma 4.1], shows that the infimum in Eq. (3.13) is attained.

Lemma 3.19. Given any connected weighted graph $G = (V, w)$, there exist elements α, γ of I^n with $0 < \alpha + \gamma \leq 1$ such that $\beta_{W_G} = \tilde{\beta}_G(\alpha, \gamma)$.

Proof. If $\beta_{W_G} = \frac{1}{2}$, then we have $\beta_{W_G} = \tilde{\beta}_G(\alpha, \gamma)$ for $\alpha = \gamma = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in I^n$.

Now assume that $\beta_{W_G} \neq \frac{1}{2}$, that is, $\beta_{W_G} < \frac{1}{2}$, by Lemma 3.7. Then, using Lemma 3.18, given any positive integer k, there exist elements $\alpha^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}), \gamma^{(k)} = (\gamma_1^{(k)}, \ldots, \gamma_n^{(k)})$ of I^n with $0 < \alpha^{(k)} + \gamma^{(k)} \leq 1$ such that

$$
\beta_{W_G} \leq \tilde{\beta}_G(\alpha^{(k)}, \gamma^{(k)}) < \beta_{W_G} + \frac{1}{k}. \tag{3.15}
$$

Since I^n is compact, the sequences $(\alpha^{(k)})$ and $(\gamma^{(k)})$ have convergent subsequences in I^n, which we again denote by $(\alpha^{(k)})$ and $(\gamma^{(k)})$, respectively, abusing the notation. Suppose they converge to α and γ, respectively, in I^n. Then for all $i, j \in V$, the sequences $(\alpha_i^{(k)})$ and $(\gamma_j^{(k)})$ converge to α_i and

γ_j, respectively, in I. Consequently, if $(1 \geq)\, \alpha + \gamma > 0$, then the sequence $\left(\tilde{\beta}_G(\alpha^{(k)}, \gamma^{(k)})\right)$ converges to $\tilde{\beta}_G(\alpha, \gamma)$ in $\mathbb{R}$. Then, using Eq. (3.15), it follows that $\beta_{W_G} = \tilde{\beta}_G(\alpha, \gamma)$. We now show that the case $\alpha + \gamma = 0$ is not possible.

If $\alpha + \gamma$ is 0, then there is a positive integer N such that for all $k \geq N$ and $j \in V$, we have $\alpha_j^{(k)} + \gamma_j^{(k)} < \delta$, where $\delta = \frac{1}{2} - \beta_{W_G}$. Hence, for all $k \geq N$, we get

$$\begin{aligned}
&\sum_{i,j \in V} [2\alpha_i^{(k)}\alpha_j^{(k)} + 2\gamma_i^{(k)}\gamma_j^{(k)} + (\alpha_i^{(k)} + \gamma_i^{(k)})(1 - (\alpha_j^{(k)} + \gamma_j^{(k)}))] w_{ij} \\
&\geq \sum_{i,j \in V} (\alpha_i^{(k)} + \gamma_i^{(k)})(1 - (\alpha_j^{(k)} + \gamma_j^{(k)})) w_{ij} \\
&\geq (1 - \delta) \sum_{i,j \in V} (\alpha_i^{(k)} + \gamma_i^{(k)}) w_{ij},
\end{aligned}$$

which implies that $\tilde{\beta}_G(\alpha^{(k)}, \gamma^{(k)}) \geq \frac{1-\delta}{2}$, for all $k \geq N$. Combining this with Eq. (3.15) gives

$$\frac{1-\delta}{2} = \frac{1}{4} + \frac{\beta_{W_G}}{2} < \beta_{W_G} + \frac{1}{k},$$

equivalently, $k < \frac{4}{1 - 2\beta_{W_G}}$ for all $k \geq N$, which is impossible. $\square$

In the following lemma, we compare the bipartiteness ratios of graphs and the associated graphons, using certain "suitable" random variables. The analogous result for the Cheeger constants is discussed in [KM24, Lemma 4.4].

Lemma 3.20. For every loopless, connected weighted graph $G = (V, w)$, the following inequality holds.

$$\frac{1}{4}\beta_G \leq \beta_{W_G} \leq \beta_G.$$

Proof. It is clear from Lemma 3.18 and Eq. (3.14) that $\beta_{W_G} \leq \beta_G$. We proceed to prove the other inequality. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be elements of I^n with $0 < \alpha + \gamma \leq 1$ satisfying $\beta_{W_G} = \tilde{\beta}_G(\alpha, \gamma)$. The existence of such elements is guaranteed by Lemma 3.19.

Let $L_1, \ldots, L_n$ and $R_1, \ldots, R_n$ be independent random variables on some probability space $(\Omega, \mathcal{A}, P)$ such that $P(L_i^{-1}(1)) = \alpha_i$, $P(L_i^{-1}(0)) = 1 - \alpha_i$,

$P(R_i^{-1}(1)) = \gamma_i$, and $P(R_i^{-1}(0)) = 1 - \gamma_i$, for all $1 \leq i \leq n$. Define random variables X and Y as follows.

$$X = \sum_{i,j \in V} [2L_iL_j + 2R_iR_j + (L_i + R_i)(1 - L_j - R_j)]w_{ij},$$

and

$$Y = 2\sum_{i,j \in V} (L_i + R_i)w_{ij}.$$

Then, since the graph G is loopless, the expectations of X and Y are

$$E[X] = \sum_{i,j \in V} [2\alpha_i\alpha_j + 2\gamma_i\gamma_j + (\alpha_i + \gamma_i)(1 - \alpha_j - \gamma_j)]w_{ij},$$

and

$$E[Y] = 2\sum_{i,j \in V} (\alpha_i + \gamma_i)w_{ij}.$$

We will now show that the inequality $X(\omega) \geq \frac{1}{4}\beta_G Y(\omega)$ holds for all $\omega \in \Omega$. Let $\omega \in \Omega$ be arbitrary. Consider the set $S = \{i \in V : L_i(\omega) = R_i(\omega) = 1\}$. If S is the empty set, then we get $X(\omega) \geq \beta_G Y(\omega)$ from Eq. (3.14), and we are done. Suppose that the set S is nonempty. Denote by $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the elements of I^n defined by

$$x_i = L_i(\omega) \quad \text{and} \quad y_i = \begin{cases} 0 & \text{if } i \in S, \\ R_i(\omega) & \text{if } i \notin S, \end{cases}$$

for all $1 \leq i \leq n$. Note that x and y are elements of $\{0,1\}^n$, and they satisfy $0 < x + y \leq 1$. So, thanks to Eq. (3.14), it suffices to prove that $X(\omega) \geq \frac{1}{4}\tilde{\beta}_G(x,y)Y(\omega)$. Observe that

$$\begin{aligned} X(\omega) = &\sum_{i,j \notin S} (2L_iL_j + 2R_iR_j + (L_i + R_i)(1 - L_j - R_j))(\omega)w_{ij} \\ &+ \sum_{i \notin S, j \in S} (2L_iL_j + 2R_iR_j + (L_i + R_i)(1 - L_j - R_j))(\omega)w_{ij} \\ &+ \sum_{i \in S, j \notin S} (2L_iL_j + 2R_iR_j + (L_i + R_i)(1 - L_j - R_j))(\omega)w_{ij} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i\in S, j\in S} (2L_iL_j + 2R_iR_j + (L_i+R_i)(1-L_j-R_j))(\omega)w_{ij} \\
&= \sum_{i,j\notin S} (2L_iL_j + 2R_iR_j + (L_i+R_i)(1-L_j-R_j))(\omega)w_{ij} \\
&\quad + \sum_{i\notin S, j\in S} (L_i+R_i)(\omega)w_{ij} + \sum_{i\in S, j\notin S} 2w_{ij} + \sum_{i\in S, j\in S} 2w_{ij} \\
&\geq \sum_{i,j\notin S} [2x_ix_j + 2y_iy_j + (x_i+y_i)(1-x_j-y_j)]w_{ij} \\
&\quad + \frac{1}{2}\sum_{i\notin S, j\in S} 2x_iw_{ij} + \sum_{i\in S, j\notin S} (2x_j + 1 - x_j - y_j)w_{ij} + \sum_{i\in S, j\in S} 2w_{ij} \\
&\geq \frac{1}{2}\sum_{i,j\in V} [2x_ix_j + 2y_iy_j + (x_i+y_i)(1-x_j-y_j)]w_{ij},
\end{aligned}
$$

and that

$$
\begin{aligned}
Y(\omega) &= 2\sum_{i\notin S, j\in V} (L_i+R_i)(\omega)w_{ij} + 2\sum_{i\in S, j\in V} (L_i+R_i)(\omega)w_{ij} \\
&= 2\sum_{i\notin S, j\in V} (x_i+y_i)w_{ij} + 2\sum_{i\in S, j\in V} 2(x_i+y_i)w_{ij} \\
&\leq 2\cdot 2\sum_{i,j\in V} (x_i+y_i)w_{ij}.
\end{aligned}
$$

Thus, we arrive at the inequality

$$X(\omega) \geq \frac{1}{4}\tilde{\beta}_G(x,y)Y(\omega) \geq \frac{1}{4}\beta_G Y(\omega).$$

As this is true for all $\omega \in \Omega$, it implies that $E[X] \geq \frac{1}{4}\beta_G E[Y]$, that is,

$$\frac{E[X]}{E[Y]} = \beta_{W_G} \geq \frac{1}{4}\beta_G,$$

using the fact that $E[Y]$ is positive. □

Remark 3.21. Given any loopless, connected weighted graph G, combining Theorem 3.1, Lemma 3.16 and Lemma 3.20 yields the inequality

$$\frac{\beta_G^2}{32} \leq 2 - \lambda_G^{\max} \leq 2\beta_G. \tag{3.16}$$

Define the linear operator $T\colon \ell^2(V) \to \ell^2(V)$ by $(Tf)(i) = \sum_{j\in V} w_{ij} f_j$. Then the Laplacian Δ_G of G is same as the operator L defined in Chapter 1. Further, let $\mathcal{S}_1$ denote the set of all nonzero functions from V to $\{-1, 0, 1\}$, and $\mathcal{S}_2$ denote the set

$$\{(A, B) \subseteq V \times V : A \cap B = \emptyset, A \cup B \neq \emptyset\}.$$

Then the function which assigns every function f in $\mathcal{S}_1$ to the ordered pair (A, B) in $\mathcal{S}_2$, where $A = f^{-1}(-1)$ and $B = f^{-1}(1)$, is a bijection satisfying the equation

$$\frac{\sum_{i,j\in V} w_{ij}(f(i) + f(j))^2}{4\sum_{i\in V} \mathrm{vol}(i) f(i)^2} = \frac{2e_G(A, A) + 2e_G(B, B) + e_G(A \cup B, (A \cup B)^c)}{2e_G(A \cup B, V)}.$$

Hence, using the definiton of β_G and Eq. (1.4), it follows that $\beta_G = \beta_T$. So, (3.16) is the dual Cheeger–Buser inequality for graphs, up to a multiplicative constant.

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